An arithmetic remark on entire periodic functions

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For every positive number \( \omega \), there exists an odd entire transcendental function

\[
f(z) = \sum_{h=0}^{\infty} a_h \frac{z^{2h+1}}{(2h+1)!}
\]

with rational integral coefficients \( a_h \) such that

\[
f(z+\omega) = f(z).
\]

1.

Denote by

\[
g(z) = \sum_{h=0}^{\infty} c_h \frac{z^{2h+1}}{(2h+1)!}
\]

an odd entire function with real coefficients \( c_h \) where, in particular,

\[
c_0 \geq 2.
\]

The odd powers of \( g(z) \) allow the similar developments

\[
\frac{g(z)^{2n+1}}{(2n+1)!} = \sum_{h=n}^{\infty} c_{nh} \frac{z^{2h+1}}{(2h+1)!} \quad (n = 0, 1, 2, \ldots),
\]

and here

\[
(1) \quad c_{nn} \geq 2^{2n+1} \quad (n = 0, 1, 2, \ldots).
\]

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Next let

\[ f(z) = \sum_{n=0}^{\infty} b_n \frac{g(z)}{(2n+1)!} \]

where \( b_0, b_1, b_2, \ldots \) denote real numbers which are determined by the following construction.

We have

\[ f(z) = \sum_{n=0}^{\infty} b_n \sum_{h=0}^{\infty} c_{nh} \frac{z^{2h+1}}{(2h+1)!} = \sum_{h=0}^{\infty} a_h \frac{z^{2h+1}}{(2h+1)!}, \]

say, and here the new coefficients \( a_h \) are given by

\[ a_h = \sum_{n=0}^{\infty} b_n c_{nh} \quad (h = 0, 1, 2, \ldots). \]

It is thus possible to choose the coefficients \( b_n \) successively such that

\[ 0 < b_0 \leq 2^{-1}, \quad \text{and} \quad a_0 \leq 1 \quad \text{is an integer}, \]

and that for \( n \geq 1 \), on account of (1),

\[ 0 \leq b_n \leq 2^{-(2n+1)}, \quad \text{and} \quad a_n \neq 0 \quad \text{is an integer}. \]

By this construction, \( f(z) \) becomes an entire transcendental function of \( z \). On putting

\[ M(r) = \max_{|z|=r} |f(z)|, \quad M_1(r) = \max_{|z|=r} |g(z)|, \]

by (2),

\[ M(r) \leq \sum_{n=0}^{\infty} 2^{-(2n+1)} \frac{M_0(r)^{2n+1}}{(2n+1)!} \]

and therefore

\[ M(r) < \exp(M_0(r)/2). \]
2.

In the result so obtained, choose now

\[ g(z) = \sin(2\pi z/\Omega), \]

where \( \Omega \) is a constant satisfying

\[ 0 < \Omega \leq \pi. \]

Then \( g(z) \) is an odd entire function with the period \( \Omega \),

\[ g(z + \Omega) = g(z), \]

and it has a power series

\[ g(z) = \sum_{n=0}^{\infty} c_n \frac{z^{2n+1}}{(2n+1)!}, \]

where \( c_0 = 2\pi/\Omega \geq 2 \) as required. The preceding construction leads therefore to an odd entire transcendental function

\[ f(z) = \sum_{n=0}^{\infty} b_n \frac{\sin(2\pi z/\Omega)^{2n+1}}{(2n+1)!}, \]

of period \( \Omega \), and with non-vanishing integral coefficients \( a_n \). The maximum modulus \( M(r) \) of this function evidently satisfies the inequality

\[ M(r) < \exp\left(\frac{2\pi r/\Omega}{2}\right); \]

for by the choice of \( g(z) \),

\[ M_1(r) < e^{2\pi r/\Omega}. \]

3.

The following result can now be proved.

**Theorem.** Let \( \omega \) be an arbitrary positive constant. There exist two positive constants \( c \) and \( r_0 \) and an odd entire transcendental function \( f(z) \) of period \( \omega \),

\[ f(z + \omega) = f(z), \]
such that the coefficients \( a_n \) in

\[
f(z) = \sum_{n=0}^{\infty} a_n \frac{z^{2n+1}}{(2n+1)!}
\]

are rational integers not zero, and that further

\[
|f(z)| < e^{\varepsilon} |z| \quad \text{if} \quad |z| \geq r_0.
\]

Proof. The assertion has already been established if \( 0 < \omega \leq \pi \). If, however, \( \omega > \pi \), then choose for \( k \) so large a positive integer that the quantity \( \Omega = \omega/k \) satisfies the inequality \( 0 < \Omega \leq \pi \). The theorem is then valid with \( \Omega \) instead of \( \omega \); but a function of period \( \Omega \) has also the period \( \omega = k\Omega \).

The interest of the theorem lies in the fact that all the function values

\[
f(\tau)(\lambda \omega), \quad \begin{cases} \lambda = 0, 1, 2, \ldots \\ \tau = 0, 1, 2, \ldots \end{cases}
\]

are rational integers. It is implicit in a theorem by Schneider [1, p. 49, Satz 12] that an entire transcendental function of bounded order and of period \( \omega \) cannot have this property.

A similar proof allows to show that there exists an entire function \( G(z) \) such that the function

\[
F(z) = \frac{e^{G(z)}}{\Gamma(z)}
\]

and all its derivatives assume rational integral values at all integral points.
Reference


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