An elementary existence theorem for entire functions

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It is proved that, for any \( m \) given distinct real numbers \( a_1, \ldots, a_m \), there exist transcendental entire functions \( f(z) \) at most of order \( m \) for which all the values
\[
f^{(n)}(a_k) \quad \{n = 0, 1, 2, \ldots\}
\]
\[
ak \quad \{k = 1, 2, \ldots, m\}
\]
are rational integers.

1.

Let \( a_1, \ldots, a_m \), where \( m \geq 2 \) (the case \( m = 1 \) is trivial), be infinitely many given distinct real numbers, and let
\[
a_{h,j} \quad \{h = 0, 1, 2, \ldots\}
\]
\[
j = 1, 2, \ldots, m\}
\]
be infinitely many real numbers still to be selected. Put
\[
g(z) = (z-a_1) \ldots (z-a_m), \quad A_k = |g'(a_k)| = \prod_{j=1}^{m} |a_k-a_j|,
\]
so that all \( A_k \) are positive numbers. Let further
\[
g_{h,j}(z) = \frac{a_{h,j}}{z-a_j} \cdot \frac{g(z)^{h+1}}{h!(h+1)!} \quad \{h = 0, 1, 2, \ldots\}
\]
\[
j = 1, 2, \ldots, m\}
\]
and

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\[ f(z) = \sum_{h=0}^{\infty} \sum_{j=1}^{m} g_{hj}(z). \]

Then, for all non-negative integers \( n \),

\[ g_{hj}^{(n)}(a_k) = 0 \text{ if } j = k \text{ and } h > n, \text{ or if } j \neq k \text{ and } h \geq n, \]

but

\[ g_{nk}^{(n)}(a_k) = a_{nk} \prod_{j=1, j \neq k}^{m} (a_k - a_j)^{n+1} / (n+1)! = \frac{a_{nk} A_k^{n+1}}{(n+1)!^{m-1}}. \]

It follows therefore that

\[ f^{(n)}(a_k) = \frac{a_{nk} A_k^{n+1}}{(n+1)!^{m-1}} + \sum_{h=0}^{n-1} \sum_{j=1}^{m} g_{hj}^{(n)}(a_k) \text{ for } n = 0, 1, 2, \ldots. \]

Here, in the double sum on the right-hand side, there occur only coefficients \( a_{hj} \) with \( 0 \leq h \leq n-1 \). This basic equation (1) enables us therefore to select the coefficients \( a_{hj} \) suitably by induction on \( h \), as follows.

Firstly, take

\[ a_{0k} = \frac{1}{A_k} (k = 1, 2, \ldots, m), \]

so that

\[ f(a_k) = a_k (k = 1, 2, \ldots, m). \]

Secondly, let \( n \geq 1 \), and assume that all coefficients \( a_{hj} \) with \( 0 \leq h \leq n-1 \) have already been fixed. There exist then, for each suffix \( k = 1, 2, \ldots, m \), just two real values of \( a_{hj} \) such that simultaneously

\[ -(n+1)!^{m-1} \leq a_{nk} A_k^{n+1} \leq + (n+1)!^{m-1}, \quad a_{nk} \neq 0, \]

and
\( f^{(n)}(a_k) \) is a rational integer.

With the coefficients \( a_{hj} \) so chosen, we find for \( f(z) \) the upper estimate

\[
|f(z)| \leq \sum_{h=0}^{\infty} \sum_{j=1}^{m} \frac{|g(z)|}{A_j z - a_j^h} \frac{|g(z)|}{A_j^h h!},
\]

which is equivalent to

\[
|f(z)| \leq \sum_{j=1}^{m} \frac{|g(z)|}{A_j z - a_j} \exp \left( \frac{|g(z)|}{A_j} \right).
\]

This estimate shows that the series for \( f(z) \) converges absolutely and uniformly in every bounded set of the complex plane and defines an entire function of \( z \) at most of order \( m \).

In fact, since there are always two choices for each of the coefficients \( a_{hj} \), we obtain a non-countable set of such functions \( f(z) \). Hence, amongst these functions, there are also non-countably many which are not polynomials and hence are transcendental entire functions. The following result has thus been established.

**THEOREM.** Let \( a_1, \ldots, a_m \) be finitely many distinct real numbers where \( m \geq 2 \). There exist non-countably many entire transcendental functions \( f(z) \) at most of order \( m \) such that all the values

\[
f^{(n)}(a_k) \quad \begin{cases} n = 0, 1, 2, \ldots \\ k = 1, 2, \ldots, m \end{cases}
\]

are rational integers.

3.

Two interesting questions arise now which I have not been able to solve. The first one concerns the extension of the theorem to the case of infinite sequences.

**PROBLEM A.** Let \( S = \{a_k\} \) be an infinite sequence of distinct real numbers without finite limit points. Which conditions has \( S \) to satisfy if there is to exist at least one entire function \( f(z) \) not a constant
such that all the values
\[ f^{(n)}(a_k) \quad \begin{cases} n = 0, 1, 2, \ldots \\ k = 1, 2, 3, \ldots \end{cases} \]
are rational integers?

In the special case when \( S \) consists of the integral multiples of a fixed positive number, I have proved that there do exist entire functions with this property; see [1].

To formulate a second problem, let again \( a_1, \ldots, a_m, \ m \geq 2 \), be a finite set of distinct real numbers, and let \( f(z) \) be one of the functions the existence of which has been established in the theorem. Since we may replace \( z \) by \( z - a_m \), there is no loss of generality in assuming that \( a_m = 0 \). With this choice, the set \( \{a_1, \ldots, a_{m-1}\} \) has then non-countably many possibilities. On the other hand, it is easily seen that there are only countably many entire functions of the form
\[
f(z) = \sum_{h=0}^{\infty} f_h \frac{z^h}{h!}
\]
with rational integral coefficients \( f_h \) which satisfy algebraic differential equations. Taking \( m = 2 \), we arrive therefore at the following question.

**PROBLEM B.** For which real values of the number \( a_1 \neq 0 \) does there exist an entire transcendental function \( f(z) \) which

(i) satisfies an algebraic differential equation, and

(ii) has the property that all the values
\[ f^{(n)}(0) \text{ and } f^{(n)}(a_1) \quad (n = 0, 1, 2, \ldots) \]
are rational integers?

Such functions always exist when \( a_1 \) is a rational multiple of \( \pi \); but I do not know whether this is the only case.
Reference


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