On a Class of Transcendental Decimal Fractions

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Almost forty years ago, I proved (cf. [1], [2]) that the decimal fraction

0.123456789101012 \cdots

is transcendental. In the present paper, this result will be generalized, as follows.

Denote by \( \alpha(n) \) an arbitrary positive integral-valued arithmetic function. Write successively after the decimal point

- each of the 1-digit numbers 1, 2, \cdots, 9, each \( \alpha(1) \) times repeated,
- each of the 2-digit numbers 10, 11, \cdots, 99, each \( \alpha(2) \) times repeated,
- each of the 3-digit numbers 100, 101, \cdots, 999, each \( \alpha(3) \) times repeated,

and so on. It will be proved that the resulting decimal fraction is transcendental.

Since this slight generalization makes perhaps the method of proof a little clearer, we establish the analogous theorem for fractions to an arbitrary integral basis \( q \geq 2 \).

§1. Let \( q \geq 2 \) be a fixed integer and put

\[ x = 1/q. \]

If

\[ a = \{a_1, a_2, a_3, \cdots \} \]

denotes a fixed sequence of positive integers, put

\[ a(0) = 0, \quad a(n) = a_1 + a_2 + \cdots + a_n \quad \text{for} \quad n \geq 1, \]

and for \( n \geq 1, \)

\[ A_n = \sum_{h=1}^{n} h(a(q^h - 1) - a(q^{h-1} - 1)) \]

\[ = na(q^{n} - 1) - (a(q - 1) + a(q^2 - 1) + \cdots + a(q^{n-1} - 1)). \]

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By digits to the basis \( q \) we mean any one of the numbers \( 0, 1, 2, \ldots, q-1 \). Instead of the ordinary decimal expansion we shall be concerned with expansions to the basis \( q \), and we shall write symbolically

\[
d_{-m}d_{-m+1} \cdots d_{-1}d_0 \cdot d_1d_2d_3 \cdots \quad \text{for} \quad \sum_{h=-m}^{\infty} d_hx^h;
\]

here the \( d_h \) denote digits.

\( \S2. \) Let in particular

\[
\sigma(a) = 0 \cdot d_1d_2d_3 \cdots
\]

be the expansion to the basis \( q \) in which we have written successively after the point the expansions to the basis \( q \) of the integers

1. \( a_1 \) times repeated,
2. \( a_2 \) times repeated,
3. \( a_3 \) times repeated,

etc.; thus \( d_1, d_2, d_3, \ldots \), are the resulting digits of \( \sigma(a) \). We begin by writing this number as a rapidly convergent series.

In their expansions to the basis \( q \), the integers from 1 to \( q-1 \) have exactly one digit, those from \( q \) to \( q^2-1 \) have exactly two digits, those from \( q^2 \) to \( q^3-1 \) have exactly three digits, etc. Hence in the expansion of \( \sigma(a) \) there are after the point

\[
a_1+a_2+\cdots+a_{q-1}=a(q-1)
\]

digits in sets of three which correspond to 3-digit integers; next there are

\[
2(a_q+a_{q+1}+\cdots+a_{q^2-1})=2(a(q^2-1)-a(q-1))
\]

digits in pairs which correspond to 2-digit integers; following this, there are

\[
3(a_{q^2}+a_{q^2+1}+\cdots+a_{q^3-1})=3(a(q^3-1)-a(q^2-1))
\]

digits in sets of three which correspond to 3-digit integers, etc.

By its definition in Section 1, \( A_n \) is then the total number of digits in \( \sigma(a) \) after the point derived from integers which, to the basis \( q \), have at most \( n \)
digits. Here the first integer which, to the basis \( q \), has \( n \) digits is
\[
q^{n-1} = 100 \cdots 00 ,
\]
where there are \( n - 1 \) digits 0. In the expansion of \( \sigma(a) \) it evidently occurs with the factor
\[
x^{A_{n-1} + n} .
\]

§3. Now put
\[
s_n = \sum_{k=q^{n-1}}^{q^n-1} k(x^{n(a(k-1)+1)} + x^{n(a(k-1)+2)} + \cdots + x^{na(k)}) ,
\]
\[
n = 1, 2, 3, \ldots ,
\]
and in particular,
\[
s_1 = \sum_{k=1}^{q-1} k(x^{a(k-1)+1} + x^{a(k-1)+2} + \cdots + x^{a(k)}) .
\]
The terms in \( s_n \) in the lowest and the highest powers of \( x \) are
\[
q^{n-1} \cdot x^{n(A_{n-1} + n) - n(1)} \quad \text{and} \quad (q^n - 1) \cdot x^{n(1)} ,
\]
respectively. Hence, for \( n \geq 2 \), the first and the last terms of the product
\[
x^{(A_{n-1} + n) - n(a(q^{n-1} - 1) + 1)} s_n = t_n \quad \text{say},
\]
are
\[
q^{n-1} \cdot x^{A_{n-1} + n} \quad \text{and} \quad (q^n - 1) \cdot x^{A_n} ,
\]
respectively.

It follows that \( s_1 \) is the sum of all the contributions to \( \sigma(a) \) from the 1-digit integers and similarly, for \( n \geq 2 \), \( t_n \) is the sum of all the contributions to \( \sigma(a) \) from the \( n \)-digit integers. Consequently
\[
\sigma(a) = s_1 + \sum_{n=2}^{\infty} t_n .
\]

Here, for \( n \geq 2 \), by the definition of \( A_n \) in Section 1,
\[
(A_{n-1} + n) - n(a(q^{n-1} - 1) + 1) = A_{n-1} - na(q^{n-1} - 1)
\]
\[
= -(a(q-1) + a(q^2 - 1) + \cdots + a(q^{n-1} - 1)) .
\]
Hence, if for \( n \geq 2 \) we put
\[
e(n) = a(q - 1) + a(q^2 - 1) + \cdots + a(q^{n-1} - 1),
\]
we have shown that
\[
\sigma(a) = s_1 + \sum_{n=2}^{\infty} x^{-e(n)} s_n.
\]

§4. In this expansion, the sums \( s_n \) can be replaced by more explicit expressions. As a geometric series,
\[
x^{n\{a(k-1)+1\}} + x^{n\{a(k-1)+2\}} + \cdots + x^{na(k)} = \frac{x^n}{1-x^n} (x^{na(k-1)} - x^{na(k)}),
\]
so that
\[
s_n = \frac{x^n}{1-x^n} \sum_{k=q^{n-1}}^{q^n-1} k(x^{na(k-1)} - x^{na(k)}),
\]
or equivalently,
\[
s_n = \frac{x^n}{1-x^n} \left( q^{n-1} x^{na(q^n-1)} - q^n x^{na(q^n-1)} + \sum_{k=q^{n-1}}^{q^n-1} x^{na(k)} \right).
\]

It follows therefore that
\[
\sigma(a) = \frac{x}{1-x} \left( 1 - qx^{a(q-1)} + \sum_{k=1}^{q-1} x^{a(k)} \right)
+ \sum_{n=2}^{\infty} \frac{x^{-e(n)}}{1-x^n} \left( q^{n-1} x^{na(q^n-1)} - q^n x^{na(q^n-1)} + \sum_{k=q^{n-1}}^{q^n-1} x^{na(k)} \right).
\]

In a special case, to which we now proceed, this somewhat involved formula can be further simplified.

§5. For this purpose we assume from now on that, for every positive integer \( n \), all the integers
\[
a_k, \text{ where } q^{n-1} \leq k \leq q^n - 1,
\]
have one and the same value, the value \( \alpha(n) \) say; here \( \alpha(n) \) is a positive integer-valued function of \( n \) which is not otherwise restricted.
We find now easily that

\[ a(k) = \alpha(1)k \quad \text{for} \quad 1 \leq k \leq q - 1, \]

\[ a(k) = (\alpha(1) + \alpha(2)q + \cdots + \alpha(n - 1)q^{n-2})(q - 1) + \alpha(n)(k - q^{-1} + 1) \]

for \( q^{-1} \leq k \leq q - 1, \quad n \geq 2. \)

Thus, in particular, for \( n = 1, 2, 3, \ldots, \)

\[ a(q^n - 1) = (\alpha(1) + \alpha(2)q + \cdots + \alpha(n)q^{n-1})(q - 1), \]

whence

\[ e(n) = ((n - 1)\alpha(1) + (n - 2)\alpha(2)q + \cdots + 1 \cdot \alpha(n - 1)q^{n-2})(q - 1) \quad \text{for} \quad n \geq 2. \]

The explicit expressions for \( a(k) \) imply next that

\[
\sum_{k=q^{-1}}^{q^n-1} x^{n\alpha(k)} = \sum_{k=q^{-1}}^{q^n-1} x^{n\{\alpha(1) + \alpha(2)q + \cdots + \alpha(n-1)q^{n-2}\}(q-1) + \alpha(n)(k - q^{-1} + 1)}
\]

\[ = x^{n\{\alpha(1) + \alpha(2)q + \cdots + \alpha(n-1)q^{n-2}\}(q-1)} \cdot x^{n\alpha(n)} \cdot \frac{1 - x^{n\alpha(n)(q-1)q^{-1}}}{1 - x^{n\alpha(n)}} , \]

while moreover,

\[ q^{n-1} x^{na(q^n-1)} = q^{n-1} x^{n\{\alpha(1) + \alpha(2)q + \cdots + \alpha(n-1)q^{n-2}\}(q-1)} , \]

\[ q^n x^{na(q^n-1)} = q^n x^{n\{\alpha(1) + \alpha(2)q + \cdots + \alpha(n-1)q^{n-2}\}(q-1)} \cdot x^{n\alpha(n)(q-1)q^{-1}} . \]

Therefore,

\[ q^{n-1} x^{na(q^n-1)} - q^n x^{na(q^n-1)} + \sum_{k=q^{-1}}^{q^n-1} x^{na(k)} = x^{n\{\alpha(1) + \alpha(2)q + \cdots + \alpha(n-1)q^{n-2}\}(q-1)} \]

\[ \times \left( \frac{q^{n-1} - q^{n-1} x^{n\alpha(n)} + x^{n\alpha(n)}}{1 - x^{n\alpha(n)}} - \frac{q^n - q^n x^{n\alpha(n)} + x^{n\alpha(n)}}{1 - x^{n\alpha(n)}} \right) . \]

It follows that

\[ s_n = \frac{x^n}{1 - x^n} \cdot \frac{x^{n\alpha(n)}}{1 - x^{n\alpha(n)}} \cdot x^{n\{\alpha(1) + \alpha(2)q + \cdots + \alpha(n-1)q^{n-2}\}(q-1)} \]

\[ \times \left( (q^{n\alpha(n) + n-1} - q^{n-1} + 1) - (q^{n\alpha(n) + n} - q^n + 1) x^{n\alpha(n)(q-1)q^{-1}} \right) ; \]

here we have applied the equation \( x = 1/q. \)
On substituting this value of $s_n$ in (1), we find that

$$\sigma(a) = \frac{x^{(n+1)}}{(1-x)(1-x^{n+1})} (q^{\alpha(1)} - (q^{\alpha(1)+1} - q + 1)x^{\alpha(1)(q-1)})$$

$$+ \sum_{n=2}^{\infty} \frac{x^{n\alpha(n)+n}}{(1-x^n)(1-x^{n\alpha(n)})}$$

$$\times \left( (q^{n\alpha(n)+n} - q^{n-1} + 1) - (q^{n\alpha(n)+n} - q^{n} + 1)x^{n\alpha(n)(q-1)q^{n-1}} \right)$$

$$\times x^{\{1 \cdot \alpha(1)+2 \cdot \alpha(2)q+\cdots+(n-1)\alpha(n-1)q^{n-2}\}}(q-1).$$

This formula can finally be simplified by taking together the positive part of the $n$-th term with the negative part of the $(n-1)$-st term throughout. Thus we arrive at the following simple expansion where we have once more used the fact that $x = 1/q$:

$$\sigma(a) = \frac{q^{\alpha(1)}}{(q-1)(q^{\alpha(1)} - 1)}$$

$$+ \sum_{n=1}^{\infty} \left( \frac{q^{n\alpha(n)+n} - q^{n} + 1}{(q^{n}-1)(q^{n\alpha(n)} - 1)} - \frac{q^{(n+1)\alpha(n+1)+n} - q^{n} + 1}{(q^{n+1}-1)(q^{(n+1)\alpha(n+1)} - 1)} \right)$$

$$\times q^{-(\alpha(1)+2\alpha(2)q+\cdots+n\alpha(n)q^{n-1})}(q-1).$$

In the special case when

$$\alpha(n) = 1 \quad \text{for all} \quad n,$$

this development of $\sigma(a)$ reduces to a formula which I obtained almost forty years ago in [1], [2]. (See also Nicholson [5] and my recent note [4].)

§6. From its definition in Section 2, $\sigma(a)$ is obviously an irrational number. We shall now decide whether this number is algebraic or not. At this point it is convenient to introduce some abbreviations. Put

$$u_n = \frac{q^{n\alpha(n)+n} - q^{n} + 1}{(q^{n}-1)(q^{n\alpha(n)} - 1)} - \frac{q^{(n+1)\alpha(n+1)+n} - q^{n} + 1}{(q^{n+1}-1)(q^{(n+1)\alpha(n+1)} - 1)}$$

and

$$E_n = \{\alpha(1)+2\alpha(2)q+\cdots+n\alpha(n)q^{n-1}\}(q-1),$$

and write $\sigma(a)$ as

$$\sigma(a) = \left( \frac{q^{\alpha(1)}}{(q-1)(q^{\alpha(1)} - 1)} - \sum_{k=1}^{n-1} u_k q^{-E_k} \right) - \sum_{k=n}^{\infty} u_k q^{-E_k}.$$
Let now
\[ D_n = (q - 1)(q^2 - 1) \cdots (q^n - 1) (q^{\alpha(1)} - 1)(q^{2\alpha(2)} - 1) \cdots (q^{n\alpha(n)} - 1) , \]
and
\[ B_n = D_n q^{E_n - 1} , \quad A_n = B_n \left( \frac{q^{\alpha(1)}}{(q - 1)(q^{\alpha(1)} - 1)} - \sum_{k=1}^{n-1} u_k q^{-E_k} \right) , \quad R_n = \sum_{k=n}^{\infty} u_k q^{-E_k} . \]

Then \( B_n > 0 \) and \( A_n \) are integers; \( R_n \) is a positive number, and

\[ \sigma(a) = \frac{A_n}{B_n} - R_n . \]

It follows easily from the definition of \( u_n \) that
\[ \lim_{n \to \infty} u_n = \frac{q - 1}{q} . \]

Since the numbers \( E_n \) increase sufficiently rapidly,

\[ R_n \sim (q - 1) q^{-(E_n + 1)} , \]

and \( n \) tends to infinity.

Further
\[ D_n < q^{(1 + 2 + \cdots + n) + \{\alpha(1) + 2\alpha(2) + \cdots + n\alpha(n)\}} , \]
whence, by the definition of \( E_n \),

\[ \lim_{n \to \infty} \frac{\log D_n}{E_n} = 0 . \]

\( \S 7. \) Next,

\[ \lim \inf_{n \to \infty} \frac{E_n}{E_{n-1}} \geq q . \]

For let this assertion be false. There exists then a constant \( c \) satisfying

\[ 0 < c < q \]

such that
\[ E_n \leq c E_{n-1} \]
for all sufficiently large \( n \). But then, as \( n \) increases indefinitely,

\[
E_n = O(c^n),
\]

contrary to

\[
E_n = \{ \alpha(1) + 2 \alpha(2)q + \cdots + n \alpha(n)q^{n-1} \}(q-1) \geq n \alpha(n)(q^n - q^{n-1}).
\]

Since \( q \geq 2 \), relation (6) implies that there exists an infinite strictly increasing sequence of positive integers

\[
N = \{ n_1, n_2, n_3, \cdots \}
\]

such that

\[
E_n > \frac{\xi}{3} E_{n-1} \quad \text{for} \quad n \in N.
\] (7)

Hence, by (3) and (4), for all sufficiently large \( n \in N \),

\[
0 < \left| \sigma(a) \frac{A_n}{B_n} \right| < q^{\frac{\xi}{3} E_{n-1}}.
\]

Further, for such \( n \), by the definition of \( B_n \) and by (5) and (7),

\[
B_n < q^{\frac{\xi}{3} E_{n-1}},
\] (8)

and therefore

\[
0 < \left| \sigma(a) \frac{A_n}{B_n} \right| < B_n^{-5/4}
\]

if \( n \in N \) is sufficiently large.

In this estimate, the denominator \( B_n = D_n q^{E_{n-1}} \) tends to infinity and, by (5), (7) and (8),

\[
D_n < B_n^{1/8}
\]

for all sufficiently large \( n \in N \). Moreover, the second factor \( q^{E_{n-1}} \) has only finitely many bounded prime factors.

It follows then, by Ridout's generalization of Roth's theorem ([6], see also [3]), that \( \sigma(a) \) is a transcendental number. We have thus proved the following result.
Theorem 1. If \( n \) runs over the positive integers, if, for every \( n \), \( \alpha(n) \) is a positive integer, and if in the definition of \( \sigma(a) \),

\[
a_k = \alpha(n) \quad \text{for} \quad q^{n-1} \leq k \leq q^n - 1,
\]

then \( \sigma(a) \) is transcendental.

The following result can also be proved.

Theorem 2. Under the same hypothesis as in Theorem 1, \( \sigma(a) \) is a Liouville number if and only if

\[
\sup_{n} \frac{E_n}{E_{n-1}} = \infty.
\]

The proof is not difficult and may be omitted.

We have assumed that \( \alpha(n) \) is always positive. Actually, this assumption is too restrictive, and Theorem 1 remains valid if \( \alpha(n) \) is allowed to assume the value 0 provided there are infinitely many \( n \) for which \( \alpha(n) \) is positive. Whenever \( \alpha(n) \) vanishes, the representation (2) of \( \sigma(a) \) becomes invalid. But, in order to obtain a correct representation of \( \sigma(a) \), it suffices to omit in (2) the two contributions which correspond to each \( \alpha(n) = 0 \).

There remains the unsolved problem whether Theorem 1 has an analogue for general sequences \( a \).

Bibliography


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