On a Special Function

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Over 50 years ago, when I was his student at the University of Frankfurt a.M., C. L. Siegel explained to me how to apply Mellin's integral $e^{-t} = (1/2\pi i) \times \int \Gamma(s) t^{-s} ds$, where the integration is over a line parallel to the imaginary axis and to the right of $s = 0$, to the study of the function $f(z) = \sum_{n=0}^{\infty} z^{2n}$ in the neighborhood of roots of unity on the complex unit circle $|z| = 1$. I later could obtain similar results by means of Poisson's or Euler's summation formula. In the present note I return to this old problem and obtain estimates by means of a very elementary method. It has the further advantage that it allows the study of $f(z)$ in the neighborhood of points on the unit circle which are not roots of unity.

1. Let $z$ be a complex variable. The power series

$$f(z) = \sum_{n=0}^{\infty} z^{2n}$$

converges and defines a regular function when $z$ lies in the unit disk

$$|z| < 1,$$

but it cannot be continued beyond this disk. For let

$$\epsilon = e^{2\pi i k/2^m},$$

where $m$ and $k$ are integers such that $m \geq 0$ and $0 \leq k \leq 2^m - 1$, be an arbitrary $2^m$th root of unity. Then

$$f(z) = \sum_{n=0}^{m-1} (\epsilon r)^{2n} + \sum_{n=m}^{\infty} r^{2n}$$

if $z = \epsilon r$ and $0 \leq r < 1$, and here the first sum remains bounded while the second one tends to $+\infty$ as $r$ tends to 1. Therefore all the $2^m$th roots of unity
are singular points of \( f(z) \), and since these roots of unity are everywhere dense on the unit circle \( |z| = 1 \), this circle is a natural boundary for \( f(z) \).

We shall now make this well-known result more precise by estimating how \( f(z) \) behaves when \( z \) approaches the unit circle.

2. For this purpose write \( z \) in the form

\[
z = e^{-t + \phi i},
\]

where \( t \) is a positive number and \( \phi \) a real number. We are interested in the behaviour of \( f(z) \) as \( t \), for arbitrary \( \phi \), tends to 0 and may therefore, without loss of generality, assume that already

\[
0 < t \leq 1.
\]

Let, as usual, \([x]\) denote the integral part of the real number \( x \). Then associate with \( t \) the nonnegative integer

\[
N = \left\lceil \frac{\log(1/t)}{\log 2} \right\rceil; \tag{1}
\]

hence

\[
2^N t \leq 1 < 2^{N+1} t. \tag{2}
\]

The power series \( f(z) \) can be split into the two sums

\[
f(z) = f_1(z) + f_2(z),
\]

where

\[
f_1(z) = \sum_{n=0}^{N-1} z^{2^n} \quad \text{and} \quad f_2(z) = \sum_{n=N}^{\infty} z^{2^n}.
\]

For the terms of \( f_1(z) \),

\[
z^{2^n} = e^{-2^n t} \cdot e^{2^n \phi i} = e^{2^n \phi i} + e^{2^n \phi i}(e^{-2^n t} - 1),
\]

so that

\[
|z^{2^n} - e^{2^n \phi i}| = 1 - e^{-2^n t}.
\]

Now for real \( x \),

\[
e^x \geq 1 + x. \tag{3}
\]
Therefore
\[ 1 - 2^n t \leq e^{-2^n t} \leq 1, \]
whence
\[ 0 \leq 1 - e^{-2^n t} \leq 2^n t. \]

It follows then from (2) and (3) that
\[ \left| f_1(z) - \sum_{n=0}^{N-1} e^{2^n \phi i} \right| \leq \sum_{n=0}^{N-1} 2^n t = (2^N - 1) t \leq 1. \] (4)

Next,
\[ |f_2(z)| \leq \sum_{n=N}^{\infty} e^{-2^n t} \leq \sum_{k=1}^{\infty} e^{-2^N k t} = e^{-2^N t (1 - e^{-2^N t})^{-1}} = (e^{2^N t} - 1)^{-1}, \]
where by (2) and (3),
\[ e^{2^N t} - 1 \geq 2^N t \geq 1/2. \]

It follows that
\[ |f_2(z)| \leq 2. \] (5)

On combining the estimates (4) and (5), the following result is found.

**Theorem 1.** Let \( t \) and \( \phi \) be real numbers where \( 0 < t \leq 1 \), and let \( N \) be the nonnegative integer defined by (1). Then uniformly in \( t \) and \( \phi \),
\[ \left| f(z) - \sum_{n=0}^{N-1} e^{2^n \phi i} \right| \leq 3. \] (6)

I have not tried to replace the constant 3 on the right-hand side by the best possible constant.

3. The definition (1) of \( N \) implies that
\[ N \sim \frac{\log(1/t)}{\log 2}, \]
and so it follows from (6) that
\[ \frac{\log 2}{\log(1/t)} f(e^{-t+\phi i}) = \frac{1}{N} \sum_{n=0}^{N-1} e^{2^n \phi i} + O(1/N) \] (7)
uniformly in \( t \) and \( \phi \) if \( 0 < t \leq 1 \).
This equation suggests the following notation. In general, as \( t \) tends to 0 through positive values, or equivalently, as \( N \) tends to infinity, neither the expression on the left-hand side of (7) nor the first term on the right-hand side of (7) needs tend to a unique limit. Therefore, for each fixed value of \( \phi \), denote by \( S(\phi) \) the set of all possible limits of

\[
\frac{\log 2}{\log(1/t)} f(e^{-t+i\phi})
\]

as \( t \to +0 \), and similarly by \( T(\phi) \) the set of all possible limits of

\[
\frac{1}{N} \sum_{n=0}^{N-1} e^{2n\phi i}
\]

as \( N \to \infty \). The relation between \( t \) and \( N \) ensures then that always

\[
S(\phi) = T(\phi). \tag{8}
\]

However, exceptionally it may happen that the ordinary limit

\[
\lim_{t \to +0} \frac{\log 2}{\log(1/t)} f(e^{-t+i\phi}), \quad s(\phi) \text{ say},
\]

or the ordinary limit

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{2n\phi i}, \quad t(\phi) \text{ say},
\]

does in fact exist. If this is so, then both limits exist simultaneously, and

\[
s(\phi) = t(\phi). \tag{9}
\]

The function \( f(z) \) satisfies the functional equation

\[
f(z) = f(z^2) + z.
\]

From this it follows immediately that

\[
S(2\phi) = S(\phi) \quad \text{and} \quad T(2\phi) = T(\phi), \tag{10}
\]

and if \( s(\phi) \) and \( t(\phi) \) exist, also

\[
s(2\phi) = s(\phi) \quad \text{and} \quad t(2\phi) = t(\phi). \tag{11}
\]
In particular,
\[ s(0) = t(0) = 1. \] (12)

4. It is convenient to replace \( \phi \) in the last formulas by \( 2\pi \psi \) where \( \psi \) is a further real number because the exponential function of \( \psi \)

\[ e(\psi) = e^{2\pi i \psi} \]

has the period 1. Further put

\[ S[\psi] = S(2\pi \psi), \quad T[\psi] = T(2\pi \psi), \quad s[\psi] = s(2\pi \psi), \quad t[\psi] = t(2\pi \psi), \]

so that always

\[ S[\psi] = T[\psi], \]

and that

\[ s[\psi] = t[\psi] \]

if these limits exist.

5. In the special case when \( \psi \) is a rational number, we can easily show that \( t[\psi] \) and hence also \( s[\psi] \) exist and determine their common value. Put

\[ \psi = p/q, \]

where \( p \) and \( q \) are integers such that

\[ 0 \leq p \leq q - 1, \quad (p, q) = 1. \]

If \( q \) is a power of 2, it follows from (11) that

\[ t[p/q] = 1. \] (13)

More generally, if \( q = 2^k Q \) is the product of a power of 2 times an odd integer \( Q \), by (11)

\[ t[p/q] = t[p/Q]. \] (14)

It suffices therefore to study the case when the denominator

\[ q \] is odd.
Denote by
\[ r = \phi(q) \]
Euler’s function of \( q \), so that by Euler’s theorem
\[ 2^r \equiv 1 \pmod{q}, \]
hence
\[ e(2^m p/q) = e(2^n p/q) \quad \text{if} \quad m \equiv n \pmod{q}. \]
Hence, on writing the integer \( N \) as
\[ N = Mr + m, \]
where \( M \) and \( m \) are integers such that
\[ M \geq 0 \quad \text{and} \quad 0 \leq m \leq r - 1, \]
then
\[ \sum_{n=0}^{N-1} e(2^n p/q) = M \sum_{n=0}^{r-1} e(2^n p/q) + \sum_{n=0}^{m-1} e(2^n p/q), \]
where we have used that \( e(\psi) \) has period 1. In this formula the second sum has at most \( r \) terms and so its absolute value cannot exceed \( r \). Further, as \( N \) tends to infinity, \( M/N \) has the limit \( i/r \). It follows that \( s[p/q] \) and \( t[p/q] \) exist and are given by
\[ s[p/q] = t[p/q] = \frac{1}{r} \sum_{n=0}^{r-1} e(2^n p/q), \quad (15) \]
where \( r = \phi(q) \).

The finite sum on the right-hand side of this formula, when different from zero, is a Gaussian period from the theory of cyclotomy. (See Kummer [1] and Fuchs [2].)

6. When \( \phi = 2\pi \psi \) is not a rational multiple of \( 2\pi \), \( s[\psi] \) and \( t[\psi] \) need not exist. A simple example is given by the number
\[ \psi = \sum_{n=1}^{\infty} d_n 2^{-n}, \]
where the coefficients \( d_n \) are digits 0 and 1 defined as follows. First put \( 1! = 1 \), digit \( d_1 = 1 \), then \( 2! = 2 \) pairs of digits 0, 1 so that \( d_2 = d_4 = 0, d_3 = d_5 = 1 \).
Then put again $3! = 6$ single digits 1, followed by $4! = 24$ pairs of digits 0, 1. Generally, alternate between $(2n - 1)!$ single digits 1 and $(2n)!$ pairs of digits 0, 1. It is easily seen that the two sets $S[\psi] = T[\psi]$ contain at least two distinct limit points, hence that $s[\psi]$ and $t[\psi]$ do not exist with this choice of $\psi$.

In a different direction there is a classical theorem by Borel and Weyl which states that

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} e(2^n \psi) = 0
$$

for almost all real $\psi$. Hence by (7) for almost all points $e(\psi)$ on the unit circle for approach along the radius

$$
f(e^{-t + 2\pi i \psi}) = o(\log(1/t)).
$$

In the neighborhood of the unit circle $f(z)$ oscillates violently as is clear from tabulating its values. The function has exactly one real zero $\neq 0$ at

$$-0.658 626 8,$$

and I found three pairs of complex roots

$$0.120 314 8 \pm i.0.934 605 9,$$
$$0.391 862 7 \pm i.0.898 257 6,$$
$$-0.685 206 2 \pm i.0.670 534 1.$$

It is highly probable that $f(z)$ has zeros in every neighborhood of the unit circle, but I have not proved this.

**References**