A New Transfer Principle in the
Geometry of Numbers

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general transfer principle in the geometry of numbers which consisted of inequalities
linking the successive minima of a convex body in \( n \) dimensions with those of a
convex body in \( N \) dimensions where in general \( N > n \). This result con-
tained in particular my earlier theorem on compound convex bodies (Proc. London
Math. Soc. (3) 5 (1955), 358–379). In the present paper I apply essentially the same
method to prove a new transfer principle which connects the successive minima of a
convex body in \( m \) dimensions and those of a convex body in \( n \) dimensions with
the successive minima of a convex body in \( mn \) dimensions.

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1. Let \( m \geq 2 \) and \( n \geq 2 \) be integers, let \( \mathbb{R}^m \) and \( \mathbb{R}^n \) be the real
\( m \)-dimensional and \( n \)-dimensional spaces of all points or vectors
\[
x = (x_1, \ldots, x_m) \quad \text{and} \quad y = (y_1, \ldots, y_n),
\]
respectively, and let \( \mathbb{R}^{mn} \) be the real \( mn \)-dimensional space of all points or
vectors
\[
z = (z_{11}, z_{12}, \ldots, z_{mn}),
\]
where the coordinates
\[
z_{hk}, \quad (h = 1, 2, \ldots, m, k = 1, 2, \ldots, n)
\]
are arranged in lexicographical order. We denote by
\[
u_1 = (1, \ldots, 0), \ldots, u_m = (0, \ldots, 1)
\]
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the $m$ points in $R^m$ with just one coordinate 1 and all others 0, by

$$v_1 = (1,\ldots,0), \ldots, v_n = (0,\ldots,1)$$

the analogous points in $R^n$, and by

$$w_{hk}, \quad (h = 1, 2, \ldots, m, k = 1, 2, \ldots, n)$$

the $mn$ points in $R^{mn}$ which have a coordinate 1 at the place $h, k$ and 0 at all other places. With the usual notation for sums of vectors and for the product of a vector with a scalar, the points $x$, $y$, and $z$ may then be written as

$$x = \sum_{h=1}^{m} x_h u_h, \quad y = \sum_{k=1}^{n} y_k v_k, \quad z = \sum_{h=1}^{m} \sum_{k=1}^{n} z_{hk} w_{hk}.$$

Finally, denote by $L^m$, $L^n$, and $L^{mn}$ the lattices of all points in $R^m$, $R^n$, and $R^{mn}$, respectively, which have integral coordinates. Then the lattice points $u_h$ form a basis of $L^m$, the lattice points $v_k$ a basis of $L^n$, and the lattice points $w_{hk}$ form a basis of $L^{mn}$. All three lattices have the determinant 1.

2. We introduce now the mapping $R^m \times R^n \to R^{mn}$ defined by the equations

$$z_{hk} = x_h \cdot y_k, \quad (h = 1, 2, \ldots, m, k = 1, 2, \ldots, n).$$

We write $z = x \times y$ and note that here the order of $x$ and $y$ may not be altered.

When $x$ runs over the whole space $R^m$ and $y$ over the whole space $R^n$, then $z = x \times y$ describes the algebraic manifold in $R^{mn}$, $M$ say, which is defined by the algebraic equations

$$z_{hk} z_{ij} = z_{ij} z_{hk}, \quad h, i = 1, 2, \ldots, m, k, j = 1, 2, \ldots, n).$$

Since $u_h \times v_k = w_{hk}$ for $h = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n$, the manifold $M$ contains the $mn$ unit points $w_{hk}$ which together span the space $R^{mn}$.

In the equation $z = x \times y$ the coordinates of $z$ are bilinear forms in the coordinates of $x$ and of $y$ and hence are continuous functions in these coordinates.

3. Denote by

$$A = (a_{hi}) \quad \text{and} \quad B = (b_{kj})$$
a real non-singular $m \times m$ matrix of determinant
\[ a = \det(a_{hi}) \neq 0 \]
and a real non-singular $n \times n$ matrix of determinant
\[ b = \det(b_{kj}) \neq 0. \]
We associate with $A$ the non-singular linear transformation of $\mathbb{R}^m$ defined by
\[ X = Ax = (X_1, \ldots, X_m), \quad \text{where} \quad X_h = \sum_{i=1}^{m} a_{hi}x_i \quad (h = 1, 2, \ldots, m) \]
and with $B$ the non-singular linear transformation of $\mathbb{R}^n$ defined by
\[ Y = By = (Y_1, \ldots, Y_n), \quad \text{where} \quad Y_k = \sum_{j=1}^{n} b_{kj}y_j \quad (k = 1, 2, \ldots, n). \]
If simultaneously $A$ is applied to $x$ and $B$ to $y$, then $z = x \times y$ is changed into
\[ Z = Ax \times By = X \times Y = (Z_{11}, Z_{12}, \ldots, Z_{mn}), \]
where the new coordinates $Z_{hk}$ are again numbered lexicographically and have the values
\[ Z_{hk} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{hi}b_{kj}z_{ij}, \quad (h = 1, 2, \ldots, m, k = 1, 2, \ldots, n). \]
This is again a linear transformation of $\mathbb{R}^{mn}$ defined by
\[ Z = Cz, \quad \text{where} \quad C = (c_{hi,kj}) \]
and \[ c_{hi,kj} = a_{hi}b_{kj}, \quad (h, i = 1, 2, \ldots, m, k, j = 1, 2, \ldots, n). \]
As is well known, the $mn \times mn$ matrix $C$ has the determinant
\[ c = \det(c_{hi,kj}) = a^n b^m \neq 0, \]
so that also $C$ is non-singular. We shall use the notation $C = A \times B$.

4. A “body” is a point set with interior points and a “convex body” a closed bounded convex body which is symmetric in the coordinate origin $o = (0, \ldots, 0)$, and for which $o$ is an interior point.
Let $K^m$ be any convex body in $\mathbb{R}^m$ and $K^n$ any convex body in $\mathbb{R}^n$. As the
point \( x \) runs over the whole of \( K^m \) and the point \( y \) over the whole of \( K^n \), the product point

\[ z = x \times y \]  

(1)

describes a certain point set, \( \Sigma \) say, which is a subset of the manifold \( M \). Denote by \( K^{mn} \) the convex hull of \( \Sigma \) so that \( K^{mn} \) is a convex point set in \( \mathbb{R}^{mn} \). We shall use the notation

\[ K^{mn} = K^m \times K^n. \]

**Lemma 1.** The point set \( K^{mn} \) is a convex body.

**Proof.** Since the mapping (1) is continuous, both \( \Sigma \) and \( K^{mn} \) are bounded closed point sets; further \( K^{mn} \), as already said, is convex.

Next, if \( x \) is any point of \( K^m \), then also \( -x \) belongs to \( K^m \). Now

\[ (-x) \times y = -x \times y. \]

It follows that if \( z \) is any point of \( K^{mn} \), then also \( -z \) belongs to \( K^{mn} \), and hence \( K^{mn} \) is symmetric in \( o \).

Finally, \( o \) is an interior point of \( K^{mn} \). For both \( K^m \) and \( K^n \) contain the origins of \( \mathbb{R}^m \) and of \( \mathbb{R}^n \), respectively, as interior points. This implies that there exist two positive constants \( \delta \) and \( \varepsilon \) such that \( K^m \) contains the \( 2m \) points

\[ \pm \delta \cdot u_h \quad (h = 1, 2, \ldots, m), \]

\( K^n \) contains the \( 2n \) points

\[ \pm \varepsilon \cdot v_k \quad (k = 1, 2, \ldots, n), \]

and therefore both the set \( \Sigma \) and the convex body \( K^{mn} \) contain the \( 2mn \) points

\[ \pm \delta \varepsilon \cdot w_{hk} \quad (h = 1, 2, \ldots, m, k = 1, 2, \ldots, n). \]

But then, by convexity, \( K^{mn} \) contains all points of the form

\[ \delta \varepsilon \sum_{h=1}^{m} \sum_{k=1}^{n} t_{hk} w_{hk}, \]

where \( t_{11}, t_{12}, \ldots, t_{mn} \) denote any real numbers satisfying the inequality

\[ \sum_{h=1}^{m} \sum_{k=1}^{n} |t_{hk}| \leq 1. \]
Since the $mn$ points $w_{hk}$ span the space $R^{mn}$, it follows that $o$ is an interior point of $K^{mn}$. This concludes the proof.

5. Let again $A$, $B$, and $C$ be the transformations in Section 3, and let further $K^{mn} = K^m \times K^n$. Put

$$K^{om} = AK^m, \quad K^{on} = BK^n, \quad \text{and} \quad K^{omm} = CK^{mn}.$$ 

Here $AK^m$ is to consist of all points $Ax$, where $x$ belongs to $K^m$, and similarly for $BK^n$ and $CK^{mn}$. Since we are dealing with affine transformations, $K^{om}$, $K^{on}$, and $K^{omm}$ are again convex bodies, and moreover

$$K^{omm} = K^{om} \times K^{on}.$$ 

Next denote by

$$J^{(m)} = \int_{K^m} \cdots \int dx_1 \cdots dx_m, \quad J^{(n)} = \int_{K^n} \cdots \int dy_1 \cdots dy_n,$$

$$J^{(m,n)} = \int_{K^{mn}} \cdots \int dz_{11} dz_{12} \cdots dz_{mn},$$

the volumes of $K^m$, $K^n$, and $K^{mn}$ in their respective spaces and by $J^{o(m)}$, $J^{o(n)}$, and $J^{o(m,n)}$ the analogous volumes of $K^{om}$, $K^{on}$, and $K^{omm}$, respectively. Then evidently

$$J^{o(m)} = aJ^{(m)}, \quad J^{o(n)} = bJ^{(n)}, \quad \text{and} \quad J^{o(m,n)} = cJ^{(m,n)} = a^n b^m J^{(m,n)}.$$ 

Therefore

$$J^{o(m)n} J^{o(n)m} / J^{o(m,n)} = J^{(m)n} J^{(n)m} / J^{(m,n)}$$

so that this quotient of volumes is invariant under the transformations.

6. Consider first a special case. Denote by $G^m$ and $G^n$ the unit ball $|x| \leq 1$ in $R^m$ and the unit ball $|y| \leq 1$ in $R^n$ and define a convex body $G^{mn}$ by the equation

$$G^{mn} = G^m \times G^n.$$ 

This body $G^{mn}$ is rather complicated and is in fact the convex hull of the intersection of the unit ball $|z| \leq 1$ in $R^{mn}$ with the manifold $M$. Let $g^{(m)}$, $g^{(n)}$, and $g^{(m,n)}$ be the volumes of $G^m$, $G^n$, and $G^{mn}$, respectively. These three volumes depend only on the degrees $m$ and $n$.

Next let $E^m$ be any ellipsoid in $R^m$ and $E^n$ any ellipsoid in $R^n$, both with
their centres at the origins of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and let $E^{mn}$ be the convex body in $\mathbb{R}^{mn}$ defined by

$$E^{mn} = E^m \times E^n.$$ 

The volumes of $E^m$, $E^n$, and $E^{mn}$ will be denoted by $e^{(m)}$, $e^{(n)}$, and $e^{(m,n)}$, respectively.

**Lemma 2.** There exists a positive number $c_1$ depending only on $m$ and $n$ such that

$$e^{(m,n)} = c_1 e^{(m)n} \cdot e^{(n)m}.$$ 

**Proof.** There exist two non-singular linear transformations $A$ in $\mathbb{R}^m$ and $B$ in $\mathbb{R}^n$ such that

$$E^m = AG^m \quad \text{ and } \quad E^n = BG^n$$

and therefore

$$E^{mn} = CG^{mn},$$

where $C$ is derived from $A$ and $B$ as in Section 3. It follows now from Section 5 that

$$e^{(m)} = ag^{(m)}, \quad e^{(n)} = bg^{(n)}, \quad e^{(m,n)} = cg^{(m,n)} = a^n b^m g^{(m,n)},$$

whence the assertion on putting

$$c_1 = g^{(m,n)} / g^{(m)n} g^{(n)m}.$$ 

**7.** If $S$ is any point set and $s > 0$ is a scalar, denote as usual by $sS$ the set of all points $sP$ where $P$ runs over $S$. It is obvious that in this notation, for every convex body $K^m$ in $\mathbb{R}^m$ and every convex body $K^n$ in $\mathbb{R}^n$ and for any two positive numbers $s$ and $t$, from the definition of $K^m \times K^n$,

$$sK^m \times tK^n = stK^{mn}.$$ 

By the same definition, if $K_1^m$ and $K_2^m$ are two convex bodies in $\mathbb{R}^m$, and $K_1^n$ and $K_2^n$ are two convex bodies in $\mathbb{R}^n$, such that

$$K_1^m \subset K_2^m \quad \text{ and } \quad K_1^n \subset K_2^n$$

and if further

$$K_1^{mn} = K_1^m \times K_1^n \quad \text{ and } \quad K_2^{mn} = K_2^m \times K_2^n,$$
then also

\[ K_{1}^{mn} \subset K_{2}^{mn}. \]

Let now again \( K^{m}, K^{n}, \) and \( K^{mn} = K^{m} \times K^{n} \) be the original convex bodies in \( \mathbb{R}^{m}, \mathbb{R}^{n} \), and \( \mathbb{R}^{mn} \), respectively, and let \( J^{(m)}, J^{(n)}, \) and \( J^{(m,n)} \) be their volumes. Then the following result holds:

**Theorem 1.** There exist two positive constants \( c_{2} \) and \( c_{3} \) which depend only on the dimensions \( m \) and \( n \) such that

\[ c_{2} J^{(m,n)} J^{(n)m} \leq J^{(m,n)} \leq c_{3} J^{(m,n)} J^{(n)m}. \]

**Proof.** By a theorem by John [1] there exists in \( \mathbb{R}^{m} \) an ellipsoid \( E^{m} \) and in \( \mathbb{R}^{n} \) an ellipsoid \( E^{n} \) such that

\[ m^{-1/2} E^{m} \subset K^{m} \subset E^{m} \quad \text{and} \quad n^{-1/2} E^{n} \subset K^{n} \subset E^{n}, \]

hence that

\[ (mn)^{-1/2} E^{mn} \subset K^{mn} \subset E^{mn}. \]

Let again \( J^{(m)}, J^{(n)}, J^{(m,n)}, e^{(m)}, e^{(n)}, e^{(m,n)} \) be the volume of \( K^{m}, K^{n}, K^{mn}, E^{m}, E^{n}, \) and \( E^{mn} \), respectively. Then \( m^{-1/2} E^{m} \) has the volume \( m^{-m/2} e^{(m)} \), \( n^{-1/2} E^{n} \) has the volume \( n^{-n/2} e^{(n)} \), and \( (mn)^{-1/2} E^{mn} \) has the volume \( (mn)^{-mn/2} e^{(m,n)} \). By what has already been proved,

\[ m^{-m/2} e^{(m)} \leq J^{(m)} \leq e^{(m)}, \quad n^{-n/2} e^{(n)} \leq J^{(n)} \leq e^{(n)}, \]

\[ (mn)^{-mn/2} e^{(m,n)} \leq J^{(m,n)} \leq e^{(m,n)}. \]

Therefore by Lemma 2,

\[ J^{(m,n)} / J^{(m,n)} J^{(n)m} \leq e^{(m,n)} (m^{-m/2} e^{(m)})^{-n} (n^{-n/2} e^{(n)})^{-m} \leq c_{1} (mn)^{mn} \]

and

\[ J^{(m,n)} / J^{(m,n)} J^{(n)m} \geq (mn)^{-mn/2} e^{(m,n)} / e^{(m)n} e^{(n)m} = c_{1} (mn)^{-mn/2}. \]

On putting \( c_{2} = c_{1} (mn)^{-mn/2} \) and \( c_{3} = c_{1} (mn)^{mn} \), this proves the assertion.

8. To each of the three convex bodies \( K^{m}, K^{n}, \) and \( K^{mn} \) corresponds a convex distance function, \( F^{(m)}(x) \) in \( \mathbb{R}^{m} \), \( F^{(n)}(y) \) in \( \mathbb{R}^{n} \), and \( F^{(m,n)}(z) \) in \( \mathbb{R}^{mn} \), respectively. Here, e.g., \( F^{(m)}(x) \) is defined by

\[ 0 \leq F^{(m)}(x) \leq 1 \quad \text{if and only if} \quad x \in K^{m}, \]
or more explicitly,
\[ x \in sK^m \quad \text{if} \quad |s| \geq F^{(m)}(x) \quad \text{and} \quad x \notin sK^m \quad \text{if} \quad |s| < F^{(m)}(x). \]

Further,
\[
F^{(m)}(0) = 0, \quad F^{(m)}(x) > 0 \quad \text{if} \quad x \neq 0;
F^{(m)}(sx) = |s| F(x) \quad \text{for all real } s \text{ and } x \in \mathbb{R}^m;
F^{(m)}(x_1 + x_2) \leq F^{(m)}(x_1) + F^{(m)}(x_2).
\]

Analogous properties are satisfied by the two other distance functions \(F^{(n)}(y)\) and \(F^{(m,n)}(z)\), in particular,
\[
0 \leq F^{(n)}(y) \leq 1 \quad \text{if and only if} \quad y \in K^{(n)};
0 \leq F^{(m,n)}(z) \leq 1 \quad \text{if and only if} \quad z \in K^{mn}.
\]

**Lemma 3.** If \(x \in \mathbb{R}^m\) and \(y \in \mathbb{R}^n\) and therefore \(z = x \times y \in \mathbb{R}^{mn}\), then
\[
F^{(m,n)}(z) \leq F^{(m)}(x) F^{(n)}(y).
\]

**Proof.** The assertion is obvious if \(x = 0\) or \(y = 0\) and therefore \(z = 0\). Let therefore \(x \neq 0\) and \(y \neq 0\) so that
\[
F^{(m)}(x) > 0 \quad \text{and} \quad F^{(n)}(y) > 0.
\]

On putting
\[
x^0 = F^{(m)}(x)^{-1} x \quad \text{and} \quad y^0 = F^{(n)}(y)^{-1} y,
\]
evidently \(F^{(m)}(x^0) = 1\) and \(F^{(n)}(y^0) = 1\) and therefore \(x^0 \in K^m\) and \(y^0 \in K^n\).

On defining \(z^0\) now by \(z^0 = x^0 \times y^0\),
\[
z^0 = x^0 \times y^0 = F^{(m)}(x)^{-1} F^{(n)}(y)^{-1} x \times y = F^{(m)}(x)^{-1} F^{(n)}(y)^{-1} z.
\]
Since \(x^0 \in K^m\) and \(y^0 \in K^n\), also \(z^0 \in K^{mn}\) and therefore \(F^{(m,n)}(z^0) \leq 1\). But
\[
F^{(m,n)}(z^0) = F^{(m)}(x)^{-1} F^{(n)}(y)^{-1} F^{(m,n)}(z),
\]
whence the assertion.

9. We combine the results so far obtained with Minkowski's theorem on the successive minima of a convex body in a lattice (Minkowski [4]).

This theorem will be applied three times, to \(K^m\) relative to the lattice \(L^m\).
in $\mathbb{R}^m$, to $K^n$ relative to the lattice $L^n$ in $\mathbb{R}^n$, and to $K^{mn}$ relative to the lattice $L^{mn}$ in $\mathbb{R}^{mn}$. By this theorem, there exist then

$m$ linearly independent points $x^1, \ldots, x^m$ in $L^m$,

$n$ linearly independent points $y^1, \ldots, y^n$ in $L^n$,

$mn$ linearly independent points $z^1, \ldots, z^{mn}$ in $L^{mn}$,

with the corresponding successive minima

$$
\mu_h^{(m)} = F^{(m)}(x^h), \quad (h = 1, 2, \ldots, m),
$$

$$
\mu_k^{(n)} = F^{(n)}(y^k), \quad (k = 1, 2, \ldots, n),
$$

$$
\mu_l^{(m,n)} = F^{(m,n)}(z^l), \quad (l = 1, 2, \ldots, mn),
$$

such that the following properties hold:

(i)

$$
0 < \mu_1^{(m)} \leq \mu_2^{(m)} \leq \cdots \leq \mu_m^{(m)}, \quad \frac{m!}{m!} \leq J^{(m)} \mu_1^{(m)} \cdots \mu_m^{(m)} \leq 2^m,
$$

$$
0 < \mu_1^{(n)} \leq \mu_2^{(n)} \leq \cdots \leq \mu_n^{(n)}, \quad \frac{n!}{n!} \leq J^{(n)} \mu_1^{(n)} \cdots \mu_n^{(n)} \leq 2^n,
$$

$$
0 < \mu_1^{(m,n)} \leq \mu_2^{(m,n)} \leq \cdots \leq \mu_{mn}^{(m,n)}, \quad \frac{mn!}{mn!} \leq J^{(m,n)} \mu_1^{(m,n)} \cdots \mu_{mn}^{(m,n)} \leq 2^{mn}.
$$

(ii) If $X^1, \ldots, X^m$ are $m$ linearly independent points in $L^m$, $Y^1, \ldots, Y^n$ linearly independent points in $L^n$, and $Z^1, \ldots, Z^{mn}$ $mn$ linearly independent points in $L^{mn}$, and if these points are ordered such that

$$
F^{(m)}(X^1) \leq F^{(m)}(X^2) \leq \cdots \leq F^{(m)}(X^m),
$$

$$
F^{(n)}(Y^1) \leq F^{(n)}(Y^2) \leq \cdots \leq F^{(n)}(Y^n),
$$

$$
F^{(m,n)}(Z^1) \leq F^{(m,n)}(Z^2) \leq \cdots \leq F^{(m,n)}(Z^{mn}),
$$

then

$$
F^{(m)}(x^h) \geq \mu_h^{(m)}, \quad (h = 1, 2, \ldots, m),
$$

$$
F^{(n)}(y^k) \geq \mu_k^{(n)}, \quad (k = 1, 2, \ldots, n),
$$

$$
F^{(m,n)}(z^l) \geq \mu_l^{(m,n)}, \quad (l = 1, 2, \ldots, mn).
$$

Here, in the inequalities (i), the factors $J^{(m)}$, $J^{(n)}$, and $J^{(m,n)}$ are again the
volumes of the convex bodies $K^m$, $K^n$, and $K^{mn}$, respectively. We deduce from these inequalities that the quotient

$$q = \mu_1^{(m,n)} \cdots \mu_m^{(m,n)}(\mu_1^{(m)} \cdots \mu_m^{(m)})^{-n} (\mu_1^{(n)} \cdots \mu_n^{(n)})^{-m}$$

satisfies the inequalities

$$\frac{2^{mn}}{(mn)!} (2^m)^{-n} (2^n)^{-m} \leq J^{(m,n)} f^{(m)} f^{(n)} q \leq 2^{mn} \left(\frac{2^m}{m!}\right)^{-n} \left(\frac{2^n}{n!}\right)^{-m}.$$

Here apply Theorem 1 to the quotient $J^{(m,n)} f^{(m)} f^{(n)} q$ and put

$$c_4 = (c_3mn)! 2^{mn}^{-1} \quad \text{and} \quad c_5 = (m!)^n (n!)^n (c_3 2^{mn})^{-1}.$$

We obtain then the following result:

**Lemma 4.** There exist two positive constants $c_4$ and $c_5$ which depend only on $m$ and $n$ such that

$$c_4 (\mu_1^{(m)} \cdots \mu_m^{(m)})^n (\mu_1^{(n)} \cdots \mu_n^{(n)})^m \leq \mu_1^{(m,n)} \cdots \mu_m^{(m,n)} \leq c_5 (\mu_1^{(m)} \cdots \mu_m^{(m)})^n (\mu_1^{(n)} \cdots \mu_n^{(n)})^m.$$

10. Let again $x^h (h = 1, 2, \ldots, m)$ be $m$ linearly independent points in $L^m$ and $y^k (k = 1, 2, \ldots, n)$, $n$ linearly independent points in $L^n$ at which the successive minima $\mu_h^{(m)}$ and $\mu_k^{(n)}$ are attained. Then the $mn$ product points

$$Z^{hk} = x^h \times y^k, \quad (h = 1, 2, \ldots, m, k = 1, 2, \ldots, n)$$

lie in the lattice $L^{mn}$ and, moreover, they are linearly independent. For there are two non-singular transformations $A$ and $B$ as in Section 2 such that

$$x^h = Au_h \quad (h = 1, 2, \ldots, m) \quad \text{and} \quad y^k = Bv_k \quad (k = 1, 2, \ldots, n).$$

Further, $C = A \times B$ is non-singular, and

$$Z^{hk} = Cw_{hk}, \quad (h = 1, 2, \ldots, m, k = 1, 2, \ldots, n),$$

where the $mn$ unit points $w_{hk}$ span the space $R^{mn}$.

Put

$$f_{hk}^{(m,n)} = F^{(m,n)}(Z^{hk}), \quad (h = 1, 2, \ldots, m, k = 1, 2, \ldots, n)$$

and denote for $l = 1, 2, \ldots, mn$ by $f_l^{(m,n)}$ the same quantities $f_{hk}^{(m,n)}$ ordered according to size,

$$f_1^{(m,n)} \leq f_2^{(m,n)} \leq \cdots \leq f_{mn}^{(m,n)}.$$

(2)
This ordering (which will not be unique if several of the values \( f^{(m,n)}_{hk} \) are equal) establishes thus a 1-to-1 correspondence

\[
l \leftrightarrow (h, k)
\]

between the integers \( l \) in \( 1 \leq l \leq mn \) and the pairs of integers \((h, k)\) with \( 1 \leq h \leq m, 1 \leq k \leq n \).

From property (ii) of the successive minima \( \mu^{(m,n)}_l \) and from the ordering (2) it follows that

\[
f^{(m,n)}_l = \mu^{(m,n)}_l, \quad (l = 1, 2, \ldots, mn).
\]

On the other hand, by Lemma 3,

\[
f^{(m,n)}_l = F^{(m,n)}(Z^{hk}) \leq F^{(m)}(x^h) F^{(n)}(y^k) = \mu^{(m)}_h \mu^{(n)}_k.
\]

We obtain therefore the system of \( mn \) inequalities

\[
(\text{iii}) \quad \mu^{(m,n)}_l \leq \mu^{(m)}_h \mu^{(n)}_k \text{ for } l \leftrightarrow (h, k),
\]

from which, on multiplying over all suffixes \( l \), it follows in particular that

\[
\mu^{(m,n)}_1 \cdots \mu^{(m,n)}_{mn} \leq (\mu^{(m)}_1 \cdots \mu^{(m)}_m)(\mu^{(n)}_1 \cdots \mu^{(n)}_n)^m,
\]

which is slightly better than the right-hand inequality given by Lemma 4. A valid inequality is also obtained if on the left-hand side of this formula the factor \( \mu^{(m,n)}_l \) is omitted while the right-hand side is divided by the corresponding product \( \mu^{(m)}_h \mu^{(n)}_k \), where again \( l \leftrightarrow (h, k) \). On dividing now the left-hand formula in Lemma 4 by this new inequality, it follows that

\[
(\text{iv}) \quad \mu^{(m,n)}_l \geq c_4 \mu^{(m)}_h \mu^{(n)}_k, \text{ for } l \leftrightarrow (h, k).
\]

We have so obtained the following result:

**Theorem 2.** There exists a constant \( c_4 > 0 \) depending only on \( m \) and \( n \), with the following property: Denote by \( \mu^{(m)}_h \) \((h = 1, 2, \ldots, m)\), the successive minima of the convex body \( K^m \) in \( \mathbb{R}^m \), by \( \mu^{(n)}_k \) \((k = 1, 2, \ldots, n)\), the successive minima of the convex body \( K^n \) in \( \mathbb{R}^n \), and by \( \mu^{(m,n)}_l \) \((l = 1, 2, \ldots, mn)\), the successive minima of the convex body \( K^{mn} = K^m \times K^n \) in \( \mathbb{R}^{mn} \). Let further \( p^{(m,n)}_l \) \((l = 1, 2, \ldots, mn)\), be the \( mn \) products

\[
\mu^{(m,n)}_h \mu^{(n)}_k, \quad (h = 1, 2, \ldots, m, k = 1, 2, \ldots, n)
\]

numbered in order of increasing size,

\[
p^{(m,n)}_1 \leq p^{(m,n)}_2 \leq \cdots \leq p^{(m,n)}_{mn}.
\]

Then

\[
c_4 p^{(m,n)}_1 \leq \mu^{(m,n)}_l \leq p^{(m,n)}_l \quad (l = 1, 2, \ldots, mn).
\]
Hence in particular,
\[ \gamma_4 \mu^{(m)}_1 \leq \mu^{(m,n)}_1 \leq \mu^{(m,n)}_1, \quad \gamma_4 \mu^{(m)}_n \leq \mu^{(m,n)}_m \leq \mu^{(m,n)}_n. \]

11. By means of Theorem 2 we shall finally prove a property of the successive minima of convex bodies defined by linear inequalities. The two special convex distance functions
\[ F_0^{(m)}(x) = \max(|x_1|, \ldots, |x_m|) \quad \text{and} \quad F_0^{(n)}(y) = \max(|y_1|, \ldots, |y_n|) \]
generate the convex bodies
\[ K_0^m : F_0^{(m)}(x) \leq 1 \quad \text{in } \mathbb{R}^m \quad \text{and} \quad K_0^n : F_0^{(n)}(y) \leq 1 \quad \text{in } \mathbb{R}^n, \]
which are generalized cubes of side 2 with their centres at the origin of \( \mathbb{R}^m \) and \( \mathbb{R}^n \), respectively. The product body
\[ K_0^{mn} = K_0^m \times K_0^n \]
in \( \mathbb{R}^{mn} \) is rather complicated. If \( F_0^{*(m,n)}(z) \) is its distance function, then \( K_0^{mn} \) consists of the points \( z \in \mathbb{R}^{mn} \) for which
\[ F_0^{*(m,n)}(z) \leq 1. \]
We introduce the further distance function
\[ F_0^{(m,n)}(z) = \max(|z_{11}|, |z_{12}|, \ldots, |z_{mn}|) \]
and the corresponding convex body
\[ K_0^{mn} : F_0^{(m,n)}(z) \leq 1 \quad \text{in } \mathbb{R}^{mn}, \]
which is again a generalised cube of side 2 with centre at the origin. It is easily seen that
\[ K_0^{mn} \subset K_0^{mn} \]
and therefore
\[ F_0^{(m,n)}(z) \leq F_0^{*(m,n)}(z) \quad \text{for all } z \in \mathbb{R}^{mn}. \quad \text{(1)} \]

Further, the origin \( o \) is an interior point of \( K_0^{mn} \). This implies that there is a constant \( \gamma_6 > 0 \) depending only on \( m \) and \( n \) such that all points \( z \) satisfying
\[ F_0^{(m,n)}(z) \leq 1 / \gamma_6 \]
belong to $K_{0}^{mn}$, hence that

$$K_{0}^{mn} \subset c_{6} K_{0}^{mn},$$

and therefore

$$F_{0}^{*(m,n)}(z) \leq c_{6} F_{0}^{(m,n)}(z) \quad \text{for all} \quad z \in \mathbb{R}^{mn}. \quad (II)$$

12. Denote again by

$$A = (a_{hi}) \quad \text{and} \quad B = (b_{kj})$$
a real $m \times m$ matrix and a real $n \times n$ matrix, and by

$$C = A \times B = (c_{hi,kj}), \quad \text{where} \quad c_{hi,kj} = a_{hi} b_{kj},$$

the $mn \times mn$ matrix formed from $A$ and $B$. It suffices to consider the case when all three matrices have the determinants 1,

$$a = 1, \quad b = 1, \quad c = a^{n} b^{m} = 1.$$

The four new distance functions

$$F^{(m)}(x) = F_{0}^{(m)}(Ax) \quad \text{in} \quad \mathbb{R}^{m}, \quad F^{(n)}(y) = F_{0}^{(n)}(By) \quad \text{in} \quad \mathbb{R}^{n},$$

and

$$F^{*(m,n)}(z) = F_{0}^{*(m,n)}(Cz) \quad \text{and} \quad F^{(m,n)}(z) = F_{0}^{(m,n)}(Cz) \quad \text{in} \quad \mathbb{R}^{mn}$$

define the convex bodies

$$K^{m}: F^{(m)}(x) \leq 1 \quad \text{in} \quad \mathbb{R}^{m}, \quad K^{n}: F^{(n)}(y) \leq 1 \quad \text{in} \quad \mathbb{R}^{n},$$

and

$$K^{*mn}: F^{*(m,n)}(z) \leq 1 \quad \text{and} \quad K^{mn}: F^{(m,n)}(z) \leq 1 \quad \text{in} \quad \mathbb{R}^{mn}.$$
be the successive minima of $K^m$, $K^n$, $K^{mn}$, and $K^{mn}$ in the lattices $L^m$, $L^n$, and $L^{mn}$, respectively. Further denote by
\[ z^*^l \quad \text{and} \quad z^l \quad (l = 1, 2, \ldots, mn) \]
two systems of $mn$ linearly independent lattice points in $L^{mn}$ such that
\[ \mu_i^{*(m,n)} = F^{*(m,n)}(z^*^l) \quad \text{and} \quad \mu_i^{(m,n)} = F^{(m,n)}(z^l) \quad (l = 1, 2, \ldots, mn). \]
Here, by Theorem 2, if $p_i^{(m,n)}$ has the same meaning as before,
\[ c_4 \ p_i^{(m,n)} \leq \mu_i^{*(m,n)} \leq p_i^{(m,n)} \quad (l = 1, 2, \ldots, mn). \]
Further, by property (ii) of the successive minima,
\[ F^{*(m,n)}(z^l) \geq F^{*(m,n)}(z^*^l), \quad F^{(m,n)}(z^*^l) \geq F^{(m,n)}(z^l) \quad (l = 1, 2, \ldots, mn), \]
and therefore by (III)
\[ (c_4/c_6) \ p_i^{(m,n)} \leq (1/c_6) \ \mu_i^{*(m,n)} \leq \mu_i^{(m,n)} \leq p_i^{*(m,n)} \leq p_i^{(m,n)} \quad (l = 1, 2, \ldots, mn). \]
We thus arrive at the following result:

**Theorem 3.** There exists a constant $c_7 > 0$ depending only on $m$ and $n$, with the following property: Denote by $A = (a_{hi})$ a real $m \times m$ matrix and by $B = (b_{kj})$ a real $n \times n$ matrix, and let $C$ be the $mn \times mn$ matrix
\[ C = A \times B = (c_{hi,kj}), \quad \text{where} \quad c_{hi,kj} = a_{hi} \cdot b_{kj} \quad (h = 1, 2, \ldots, m, \ k = 1, 2, \ldots, n). \]
Without loss of generality, all three matrices have the determinant 1. Let $\mu_i^{(m)}$, $\mu_k^{(n)}$, and $\mu_i^{(m,n)}$ be the successive minima of the convex distance functions
\[ F^{(m)}(x) = \max_{h = 1, 2, \ldots, m} \left| \sum_{i = 1}^{m} a_{hi} x_i \right|, \quad F^{(n)}(y) = \max_{k = 1, 2, \ldots, n} \left| \sum_{j = 1}^{n} b_{kj} y_j \right|, \]
and
\[ F^{(m,n)}(z) = \max_{h = 1, 2, \ldots, m} \left| \sum_{i = 1}^{m} \sum_{j = 1}^{n} c_{hi,kj} z_{ij} \right|, \]
respectively. Denote by $p_i^{(m,n)} \ (l = 1, 2, \ldots, mn)$ the products
\[ \mu_h^{(m)} \mu_k^{(n)} \quad (h = 1, 2, \ldots, m, k = 1, 2, \ldots, n), \]
numbered such that
\[ p_1^{(m,n)} \leq p_2^{(m,n)} \leq \cdots \leq p_{mn}^{(m,n)}. \]

Then
\[ c_7 p_l^{(m,n)} \leq \mu_i^{(m,n)} \leq p_l^{(m,n)}, \quad (l = 1, 2, \ldots, mn), \]
and in particular,
\[ c_7 \mu_1^{(m)} \mu_1^{(n)} \leq \mu_1^{(m,n)} \leq \mu_1^{(m)} \mu_1^{(n)}, \quad c_7 \mu_m^{(m)} \mu_n^{(n)} \leq \mu_{mn}^{(m,n)} \leq \mu_m^{(m)} \mu_n^{(n)}. \]

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