Alternating Projection Methods
Failure in the Absence of Convexity

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Given an initial point, x_0, sets S_i for $i = 1, \dots, r$ and their corresponding nearest point projections P_{S_i}, the *method of alternating projections (MAP)* attempts to finds a point in $\cap_{i=1}^r S_i$ by cyclically projecting onto the sets. The original alternating projection result was due to von Neumann (1933) who was able to prove that if the sets are subspaces MAP converges in norm to $P_{\cap_{i=1}^r S_i}$. Using MAP a point in the intersection can be obtained when only the individual projection onto each of the sets are known. When the underlying sets are convex, MAP and its variants are *fairly* well understood. However, despite adequate theoretical justification, these techniques are routinely applied to problems involving one or more non-convex sets with good results. In this report we examine some of the difficulties encountered when dropping the assumption of convexity. This is achieved by developing a visual tool to investigate specific examples using interactive geometry package *Cinderella* [5].

1 Preliminaries

Throughout, $\mathcal{H}$ denotes a Hilbert space equipped with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. We distinguish convergence in norm from weak convergence by $\to$ and $\xrightarrow{w}$, respectively. $P_S(x)$ denotes the nearest point projection of x onto the set S. In general this is a set valued mapping, however in the case of closed convex S the *Kolmogorov's criteria* [4, Theorem 2.8] guarantees existence and uniqueness of $P_S(x)$.

Our problem can be formulated as followed: Given an initial point $x_0 \in \mathcal{H}$ and sets $S_1, S_2, \dots, S_r$ we seek a feasible point $x^* \in \cap_{i=1}^r S_i$. This framework is sufficiently general that it applies to many cases. For example, if the sets are hyperplanes x^* solves a system of linear equations, if the sets are convex x^* solves a convex feasibility problem.

Our solution approach is the *method of alternating projections (MAP)*. Informally speaking, given an initial point we obtain each iteration by projecting onto the sets cyclically, with the desired point obtained in the limit. The method can be used to find a point in the intersection of two or more sets when it is unknown how to do so directly, but the projections onto each individual set are known. In order to guarantee convergence constraints on the underlying sets are required, typically closed subspaces or closed convex sets. The method and its variants are explained in greater detail in Section 2.

2 Three MAP Variants

In this section we present von Neumann's classical MAP, its generalisation, and two variants: Douglas-Rachford and Dykstra's method.

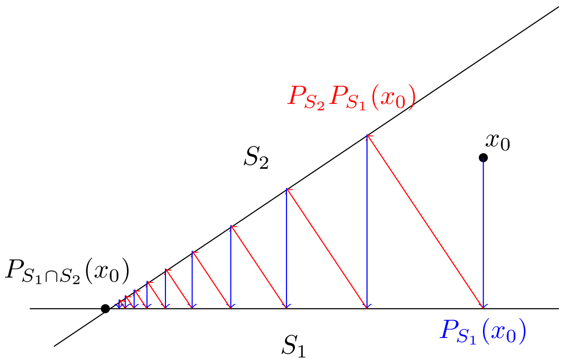

Figure 1: MAP for two 1-dimensional subspaces

Theorem 2.1 (von Neumann, 1933)**.** *Suppose S_1, S_2 are closed subspaces, then $\forall x_0 \in \mathcal{H}$:*

$$\lim_{n\to\infty} \left(P_{S_2}P_{S_1}\right)^n x_0 = P_{S_1\cap S_2}(x_0)$$

von Neumann's result was generalised to a finite number of closed subspaces by Haperin (1962). Replacing the assertion of closed subspaces with closed convex sets, Bregman (1965) was able to obtain weak convergence of the method. For closed convex sets, norm convergence remained an open question until Hundal (2002) demonstrated its failure with a counterexample. An open question is to whether MAP converges in norm under restrictions that are satisfied by many realistic cases.

Conjecture 2.2 (Norm convergence of realistic alternating projection models)**.** *If S_1 has finite codimension, closed and affine (a translate of a vector subspace) and S_2 is the nonnegative cone in $\ell_2(\mathbb{N})$, while $S_1 \cap S_2 \neq \emptyset$, the method of alternating projections is norm convergent.*

In particular, Conjecture 2.2 is known to be true in the case when S_1 is a closed hyperplane, but remains open for codimension 2.

Theorem 2.3 (Douglas-Rachford, 1959)**.** *Suppose S_1, S_2 are closed convex sets, then $\forall x_0 \in \mathcal{H}$:*

$$x_{n+1} := \frac{x_n + R_{S_2}R_{S_1}(x_n)}{2} \qquad \text{where} \qquad R_{S_i}(x) := 2P_{S_i}(x) - x$$

then $x_n \overset{w}{\rightharpoonup} x$, a fixed point, with $P_{S_1}(x) \in S_1 \cap S_2$.

Theorem 2.4 (Dysktra, 1986)**.** *Suppose S_i is a closed convex set for $i = 1, 2, \ldots, r$, then $\forall x_0 \in \mathcal{H}$:*

$$x_n^0 := x_{n-1}^r, \quad x_n^i := P_{S_i}(x_n^{i-1} - I_{n-1}^i), \quad I_n^i := x_n^i - (x_n^{i-1} - I_{n-1}^i)$$

with initial values $x_0^r := x_0$, $I_0^i := 0$, then $x_n \to P_{\cap_{i=1}^r S_i}(x_0)$.

In the particular case when the sets are subspaces, Dykstra's method reduces to the classical von Neumann MAP.

We note that many other MAP variants have been studied. Another of note is that of averaged projections, which can easily parallelised.

3 A Non-Convex Case

As a first step to developing theoretical understanding for the non-convex case, convergence results have been obtained for the Douglas-Rachford scheme in the particular two set case of a proper affine subset and a Euclidean sphere [2]. Here we consider the follow two set example, only in $\mathbb{R}^2$, with one non-convex set[1].

$$S_1 := \{\langle a, x\rangle = b : x \in \mathcal{H}\}, \qquad S_2 := \{\|x\|_{\frac{1}{2}} = 1 : x \in \mathcal{H}\}$$

The projection onto, S_1, the line (or more generally the hyperplane), P_{S_1}, can be obtained by translating the space so the hyperplane passes through the origin, projecting and then undoing the translation. This method can be used to obtain a projection whenever the sets are closed and convex [4, Exercise 5.2].

Fact 3.1 (Nearest point on hyperplane)**.** *The nearest point projection of any point $x_0 \in \mathcal{H}$ onto the hyperplane $\langle a, x\rangle = b$ is:*

$$P_{S_1}(x) = x + \frac{b - \langle a, x\rangle}{\|a\|^2} a$$

The nearest point projection onto the Euclidean sphere is simply the radial projection, $x \mapsto x/\|x\|$. However the projection on the 1/2-sphere is significantly more difficult to compute. Lagrangian multipliers can be used to obtain the projection on to the p-sphere [3, Exercise 2.3.19], with the desired result obtained by setting $p = 1/2$.

Fact 3.2 (Nearest point on p-sphere)**.** *For $0 < p < \infty$, consider the p-sphere in two dimensions:*

$$S_p := \{(x, y) : |x|^p + |y|^p = 1\}$$

Let $z^* := (1 - z^p)^{1/p}$. For $uv \neq 0$, the best approximation $P_{S_p}(u, v) = (sgn(u)z, sgn(v)z^*)$ where either $z = 0, 1$ or $0 < z < 1$ solves:

$$z^{*p-1}(z - |u|) - z^{p-1}(z^* - |v|) = 0$$

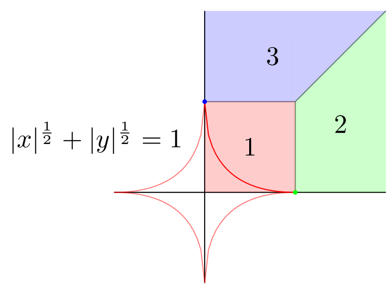

Figure 2: Important regions for the 1/2-sphere projections.

Figure 2 shows regions of interest for the projection onto the 1/2-sphere. Region 1 represent points that could potentially project onto the interior of the piecewise curve and regions 2 and 3 to the cusps at $(0, 1)$ and $(1, 0)$, respectively. For the 1/2-sphere the nearest point projection of a point x is always in the same quadrant as x.

[1] $\|x\|_p$ denotes the p-norm. In two dimensions $\|x\|_p = (|x_1|^p + |x_2|^p)^{\frac{1}{p}}$.

3.1 An Interactive Tool

To better understand MAP behaviour, a visual tool was developed using interactive geometry package *Cinderella* [5]. It is available online at http://carma.newcastle.edu.au/summer/matt/, both as a Cinderella file and as an interactive Java applet.

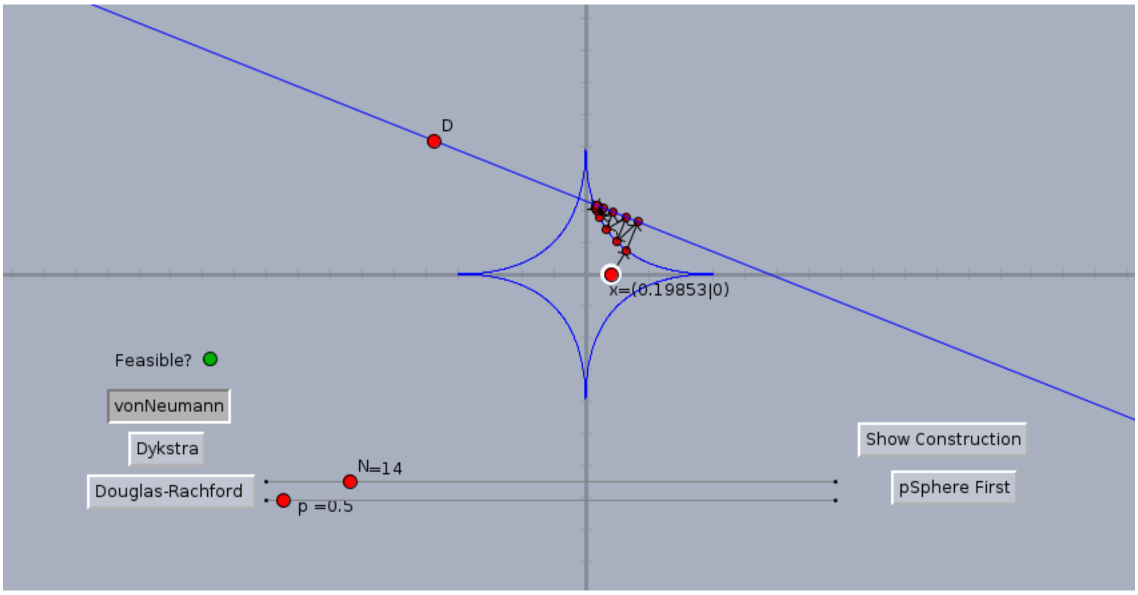

Figure 3: A screenshot of the Cinderella applet investigating an instance of von Neumann's MAP.

The applet allows for many trajectories to be rapidly investigated by moving the initial point. The feasible region can be modified by translating/rotating the line, or adjusting the p-parameter. Radio buttons on the left hand site of Figure 3 allow the projection method of choice to be selected and construction visibility can be toggled. This is particularly important for understanding behaviour. The number of iterations shown can also be varied.

3.2 Illustration of Difficulties and Results

In the case of non-convex sets, the method of alternating projections encounters a number of difficulties. Uniqueness of the nearest point projection is no longer guaranteed and consequently a point may need to be selected from the set of nearest points, at each iteration. The question *Which is the 'right' nearest point?* must be answered.

Example 3.2.1 (Non-unique nearest point)**.** *Let* $x \in \{(x,y) : y = x, x \geq 1\}$ *then* $P_{S_2}(x) = \{(1,0),(0,1)\}$.

Global convergence of the algorithm is no longer achievable, instead we aim for local convergence results. In practice, this can be troublesome, particularly in the case when a problem contains a nonempty feasible set but is 'locally infeasible' for some starting point. This is illustrated in Example 3.2.2. A potential resolution is to choose a *good* starting point, although this may be a difficult problem in itself.

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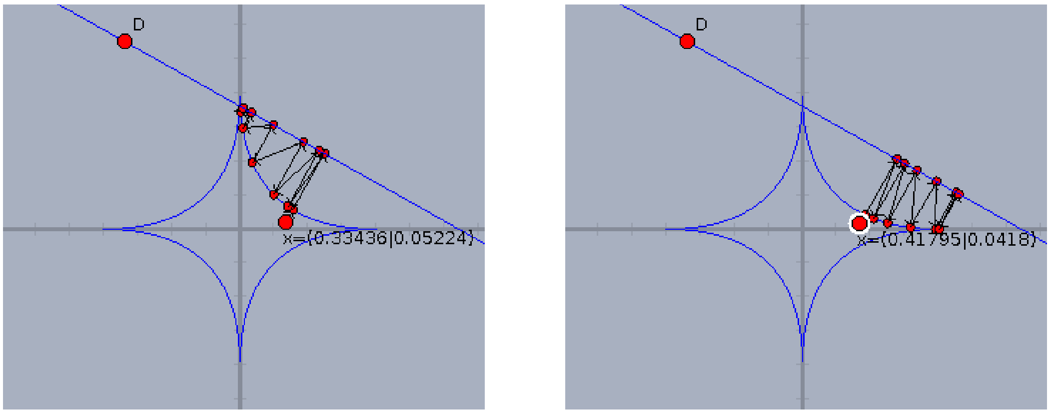

Example 3.2.2. Local convergence of von Neumann's MAP.

Proposition 3.2.1. *Consider Dykstra's MAP on two sets, where the second is the hyperplane* $\langle a, x \rangle = b$, *then for each* n, $x_n^0 - I_{n-1}^1$ *is of the form* $x_0 + ta$ *for some* $t \in \mathbb{R}$.

Proposition 3.2.1 gives a sufficient condition for failure of Dykstra's MAP. In particular, if there does not exist a sequence of points, y_n, on the line $x_0 + ta$ such that $P_{S_1}(y_n)$ approaches a feasible point. Figure 4 shows an example of this in the two dimensional case in which periodic behaviour is observed.

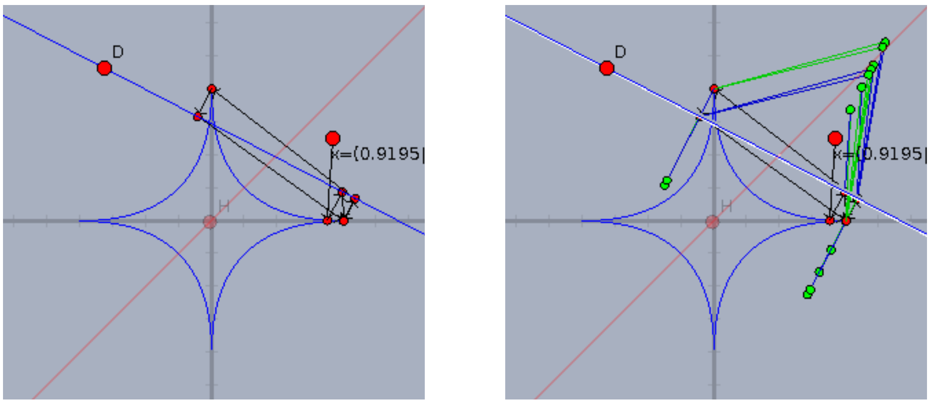

Figure 4: Periodic behaviour of Dyktra's MAP, without (left) and with (right) constructions.

This behaviour is also present using Dysktra's MAP for other non-convex cases. Figure 5 shows two sets, defined by exponential curves of the form $A + B\exp(-x/a)$, in which the iterates are conjectured to be asymptotically periodic.

Of the three MAP considered, the Douglas-Rachford scheme appears most robust. Convergence to a feasible point was always obtained. Interesting, in the case of convergence to a feasible point iterates exhibit 'spiralling behaviour' and divergence in the case of infeasibility. The latter gives a potential method to detect infeasibility.

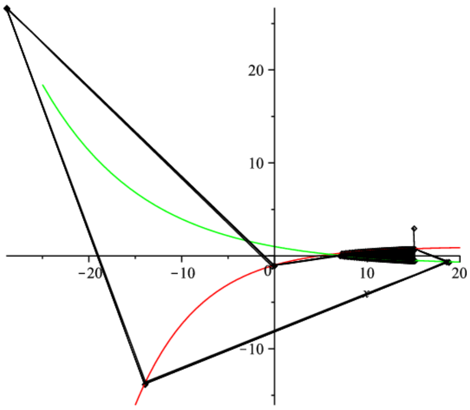

Figure 5: Conjectured asymptotic periodicity in Dykstra's MAP.

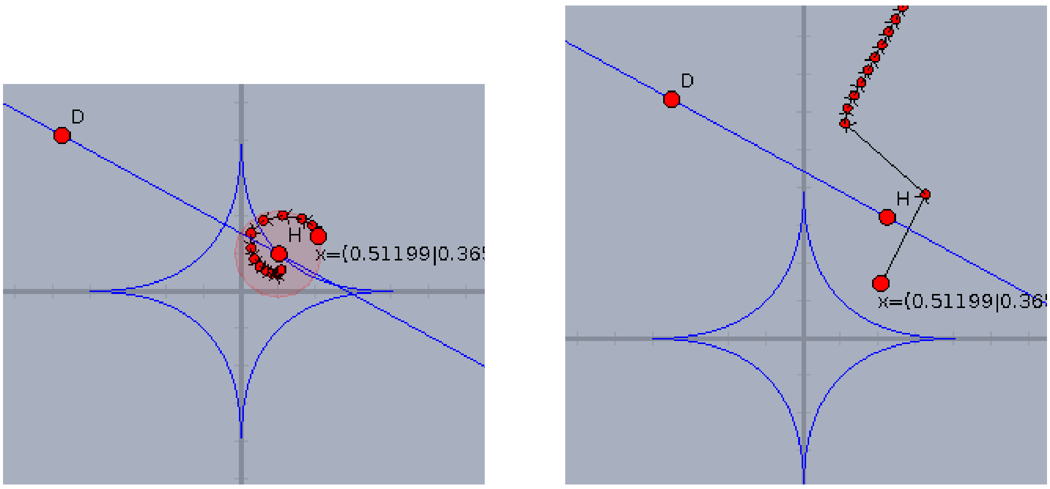

Figure 6: Behaviour of Douglas-Rachford, (left) Spiralling iterations and (right) an infeasible problem.

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