Divergence in right-angled Coxeter groups

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Outline

This is joint work with Pallavi Dani, Louisiana State University.

1. Divergence in spaces and groups
2. Right-angled Coxeter groups
3. Results and ideas of proofs
4. Subsequent work

Geodesic metric spaces

Let (X, d) be a metric space.
A geodesic segment is an isometric embedding

$$\gamma : [a, b] \to X$$

i.e. for all $a \leq s, t \leq b$,

$$d(\gamma(s), \gamma(t)) = |s - t|$$

Similarly define geodesic rays $\gamma : [a, \infty) \to X$ and geodesic lines $\gamma : (-\infty, \infty) \to X$.

(X, d) is a geodesic metric space if for all $x, y \in X$, there is a geodesic segment connecting x and y.

Examples

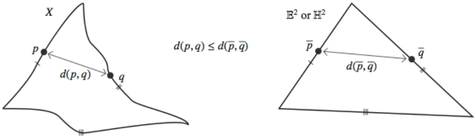
Spheres, the Euclidean plane and the hyperbolic plane are geodesic metric spaces.

Curvature conditions

Definition (Gromov)

A geodesic metric space (X, d) is

1. CAT(1) if geodesic triangles in X are "no fatter" than triangles on the sphere.
2. CAT(0) or nonpositively curved if geodesic triangles in X are "no fatter" than triangles in Euclidean space.
3. CAT(-1) or negatively curved if geodesic triangles in X are "no fatter" than triangles in hyperbolic space.



Source: Tim Riley.

If (X, d) is CAT(0) then X is contractible, uniquely geodesic, has nice boundary, finite group actions on X have fixed points, ...

One-ended spaces

A geodesic metric space is one-ended if it stays non-empty and connected when you remove arbitrarily large metric balls.

Examples

The sphere and $\mathbb{R}$ are not one-ended. For $n \geq 2$, n-dimensional Euclidean and hyperbolic space are one-ended.

Divergence of geodesics

Let (X, d) be a one-ended geodesic metric space.
Let $\gamma_1, \gamma_2 : [0, \infty) \to X$ be geodesic rays with the same basepoint.

Question

How fast do γ_1 and γ_2 move away from each other?

Definition (Gromov)

The divergence of γ_1 and γ_2 at time r is

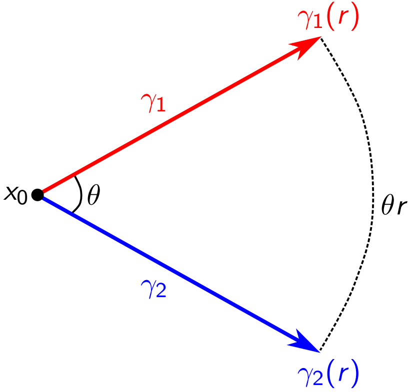
$$\operatorname{div}(\gamma_1, \gamma_2, r) := \inf_p \operatorname{length}(p)$$

where the infimum is taken over all rectifiable paths p in $X \setminus \operatorname{Ball}(x_0, r)$ connecting $\gamma_1(r)$ and $\gamma_2(r)$.

Can also define divergence of a single geodesic γ by taking $\gamma_1(r) := \gamma(r)$, $\gamma_2(r) := \gamma(-r)$ for $r \geq 0$.

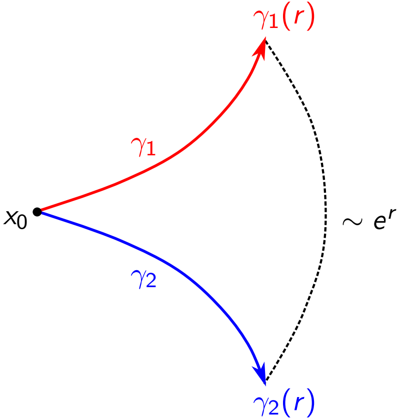
Divergence of geodesics in Euclidean space

In Euclidean space, all pairs of geodesics diverge linearly.



Divergence of geodesics in hyperbolic space

In hyperbolic space, all pairs of geodesics diverge exponentially.



Divergence of geodesics in symmetric spaces

Examples

1. In Euclidean space, all pairs of geodesics diverge linearly.
2. In hyperbolic space, all pairs of geodesics diverge exponentially.

Theorem (Gromov)

Let X be a symmetric space of noncompact type e.g. $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$. Then for all pairs of geodesics γ_1, γ_2 with common basepoint, the function $r \mapsto \mathrm{div}(\gamma_1, \gamma_2, r)$ is either linear or exponential.

Gromov asked whether the same dichotomy holds in CAT(0) spaces. It doesn't.

Divergence for finitely generated groups

Let G be a finitely generated group with finite generating set S. Let $X = \mathrm{Cay}(G, S)$. Assume X is one-ended.

Definition (Gersten 1994)

The divergence of G is the function

$$\mathrm{div}_G(r) := \sup_{x,y} \left(\inf_p \mathrm{length}(p) \right)$$

where

- the sup is over all pairs of points $x, y \in X$ at distance r from e
- the inf is over all paths p from x to y in $X \setminus \mathrm{Ball}(e, r)$.

G has linear divergence if $\mathrm{div}_G(r) \simeq r$, quadratic divergence if $\mathrm{div}_G(r) \simeq r^2$, etc, where

$$f \preceq g \iff \exists C > 0 \text{ s.t. } f(r) \leq Cg(Cr + C) + Cr + C$$

These rates of divergence are quasi-isometry invariants (Gersten).

Previous results on divergence

Many groups have divergence other than linear or exponential:

- quadratic divergence for certain free-by-cyclic groups [Gersten 1994]
- (geometric) 3-manifold groups have divergence either linear, quadratic or exponential; quadratic $\iff$ graph manifold, exponential $\iff$ hyperbolic piece [Gersten 1994, Kapovich–Leeb 1998]
- mapping class groups and Teichmüller space have quadratic divergence [Duchin–Rafi 2009]
- lattices in higher rank semisimple Lie groups conjectured to have linear divergence; proved in some cases e.g. $\mathrm{SL}(n, \mathbb{Z})$ [Drutu–Mozes–Sapir 2010]
- right-angled Artin groups have divergence linear, quadratic or exponential [Abrams–Brady–Dani–Duchin–Young 2010, Behrstock–Charney 2012]
- CAT(0) groups constructed with divergence r^d for all $d \geq 1$ [Macura 2011, Behrstock–Drutu 2011]

RAAGs and RACGs

Let Γ be a finite simplicial graph with vertex set S.

The right-angled Artin group (RAAG) associated to Γ is

$$A_\Gamma = \langle S \mid st = ts \iff s \text{ and } t \text{ are adjacent in } \Gamma \rangle$$

The right-angled Coxeter group (RACG) associated to Γ is

$$W_\Gamma = \langle S \mid st = ts \iff s \text{ and } t \text{ are adjacent in } \Gamma \text{, and } s^2 = 1 \forall s \in S \rangle$$

A_Γ and W_Γ are reducible if $S = S_1 \sqcup S_2$ with $S_i \neq \emptyset$ and $\langle S_1 \rangle$ commuting with $\langle S_2 \rangle$.

Relationship between RAAGs and RACGs

Theorem (Davis–Januszkiewicz)

Every RAAG is finite index in a RACG.

Corollary

Every RAAG is quasi-isometric to a RACG.

The converse is not true. For example A_Γ is word hyperbolic $\iff$ Γ has no edges $\iff$ A_Γ is free, but there are many word hyperbolic W_Γ which are not quasi-isometric to free groups.

Theorem (Moussong)

W_Γ is word hyperbolic if and only if Γ has no empty squares.

If W_Γ is word hyperbolic then W_Γ has exponential divergence.

Right-angled Coxeter groups

We study these groups because they

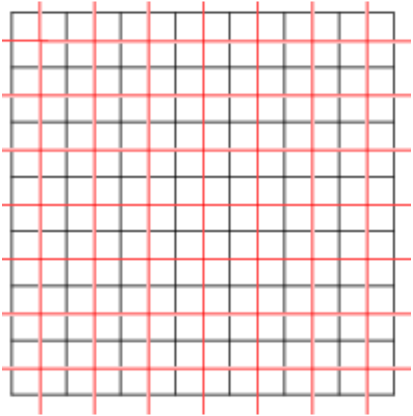
- have tractable combinatorics
- include important geometric examples
- act on nice spaces
- appear as Weyl groups for Kac–Moody Lie algebras

Examples of RACGs

1. If Γ has 2 vertices s_1, s_2 and no edges, then $W_\Gamma = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1 \rangle \cong D_\infty$ the infinite dihedral group.
2. If Γ has 2 vertices s_1, s_2 connected by an edge, then $W_\Gamma = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1 \text{ and } s_1 s_2 = s_2 s_1 \rangle \cong C_2 \times C_2$ the Klein 4-group.
3. If Γ has n vertices $s_1, \ldots, s_n$ and no edges, then W_Γ is the free product of n copies of C_2, so W_Γ has a finite index free subgroup.
4. If Γ is the complete graph on n vertices, then W_Γ is the direct product of n copies of $C_2 \iff W_\Gamma$ is finite.

Examples of RACGs

Group generated by reflections in sides of square:



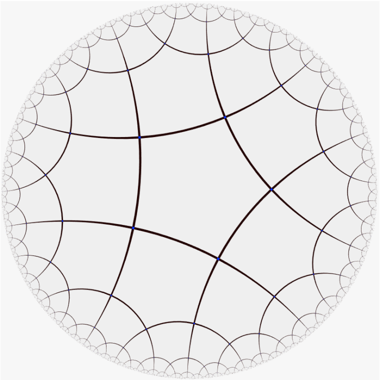
Here Γ is a 4-cycle and

$$W_\Gamma = \langle s_1, s_2, s_3, s_4 \rangle = \langle s_1, s_3 \rangle \times \langle s_2, s_4 \rangle \cong D_\infty \times D_\infty$$

The group W_Γ has linear divergence.

Examples of RACGs

Group generated by reflections in sides of right-angled hyperbolic pentagon:



The group W_Γ has exponential divergence.

Divergence in right-angled Coxeter groups

We consider W_Γ such that

- Γ is triangle-free
- Γ has no separating vertices or edges $\iff$ W_Γ is one-ended

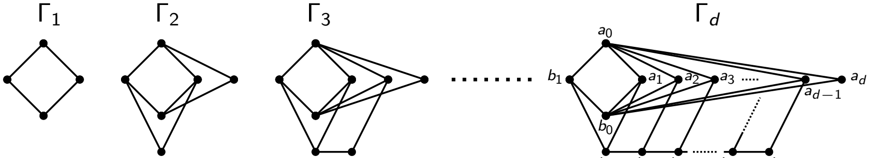
Note Γ a join $\iff$ W_Γ is reducible.

Theorem (Dani–T)

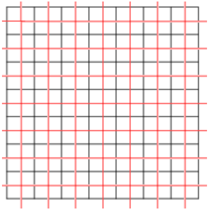
1. *W_Γ has linear divergence if and only if Γ is a join.*
2. *W_Γ has quadratic divergence if and only if Γ is $\mathcal{CFS}$ and is not a join.*

Theorem (Dani–T)

For all $d \geq 1$, the group W_{Γ_d} has divergence r^d.



The Davis complex for W_Γ



The Davis complex for a general RACG $W = W_\Gamma$ is the cube complex $\Sigma = \Sigma_\Gamma$ with

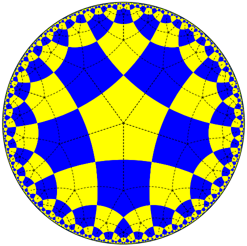
- 1-skeleton the Cayley graph of W w.r.t. S
- the cubes filled in

Theorem (Gromov)

Σ is CAT(0)*.*

W is quasi-isometric to Σ.

The Davis complex for W_Γ



Source: Jon McCammond.

The Davis complex for a general RACG $W = W_\Gamma$ is the cube complex $\Sigma = \Sigma_\Gamma$ with

- 1-skeleton the Cayley graph of W w.r.t. S
- the cubes filled in

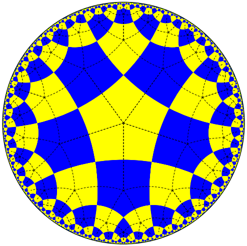
Theorem (Gromov)

Σ *is* CAT(0).

W is quasi-isometric to Σ.

Walls in the Davis complex

A reflection in W is a conjugate of a generator $s \in S$.



A wall in Σ is the fixed set of a reflection in W. We use the following properties of walls:

- walls separate Σ into two components
- length of path in $\text{Cay}(W, S) =$ number of wall-crossings
- γ is geodesic $\iff$ γ crosses each wall at most once
- walls have types $s \in S$, and walls of types s and t intersect only if $st = ts$
- walls meet at right angles in the centres of squares

4-cycles and distinguished flats

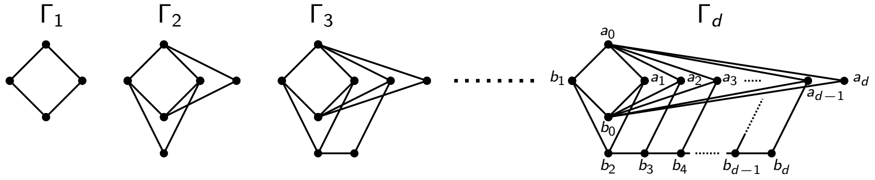
An embedded 4-cycle in Γ with vertex set $\{s, t, u, v\}$ yields the subgroup of W

$$\langle s, t, u, v\rangle = \langle s, u\rangle \times \langle t, v\rangle \cong D_\infty \times D_\infty$$

The corresponding subcomplex of Σ is a flat.

The support of a 4-cycle in Γ is the set of vertices in that 4-cycle.

If $\{s, t, u, v\}$ and $\{s, t, u, v'\}$ are both supports of 4-cycles, $v \neq v'$, then the corresponding $D_\infty \times D_\infty$ subgroups intersect along $\langle s, u\rangle \times \langle t\rangle \cong D_\infty \times C_2$. So the corresponding flats in Σ intersect along a fattened line.



The $\mathcal{CFS}$ condition

Given triangle-free Γ, form the 4-cycle graph Γ^4 with:

- vertex set the embedded 4-cycles in Γ
- two vertices adjacent if the corresponding 4-cycles have supports differing by a single vertex

That is, Γ^4 records intersections of distinguished flats along fattened lines.

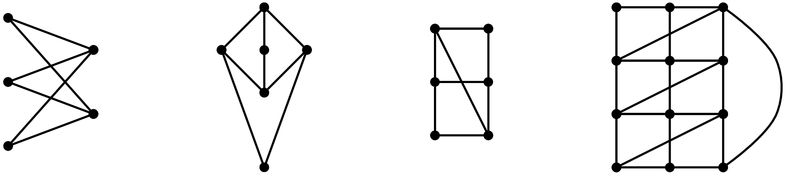
The support of a component of Γ^4 is the set of vertices of Γ, i.e. elements of S, which are in the supports of the 4-cycles in that component of Γ^4.

Definition

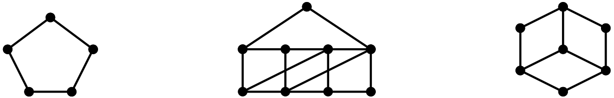
The graph Γ is $\mathcal{CFS}$ if a $\mathcal{C}$omponent of Γ^4 has $\mathcal{F}$ull $\mathcal{S}$upport.

Examples

The following graphs are $\mathcal{CFS}$:



The following graphs are not $\mathcal{CFS}$:



Characterisation of linear and quadratic divergence

We show:

1. if Γ is a join then W_Γ has linear divergence.
2. if Γ is not a join then W_Γ has divergence $\succeq r^2$.
3. if Γ is $\mathcal{CFS}$ then W_Γ has divergence $\preceq r^2$.
4. if Γ is not $\mathcal{CFS}$ and not a join then W_Γ has divergence $\succeq r^3$.

Characterisation of linear and quadratic divergence

1. *Join* $\implies$ *linear*. Direct products have linear divergence [Abrams–Brady–Dani–Duchin–Young].
2. *Not join* $\implies$ $\mathrm{div}_\Gamma \succeq r^2$. Similar to the proof for RAAGs in [ABBDY]. Since not a join, $\exists\ w = s_1 \cdots s_k$ so that s_i run through all vertices, and s_i does not commute with s_{i+1}. Consider geodesic $\gamma = w^\infty$.
3. $\mathcal{CFS}$ $\implies$ $\mathrm{div}_\Gamma \preceq r^2$. Break geodesics into pieces contained in flats coming from 4-cycles in the component of Γ^4 which has full support. Induction on number of pieces.
4. *Not* $\mathcal{CFS}$ $\implies$ $\mathrm{div}_G \succeq r^3$. Consider geodesic $\gamma = w^\infty$. Show avoidant path between $\gamma(-r)$ and $\gamma(r)$ has length at least r^3 by considering filling.

Mixture of Euclidean and hyperbolic behaviour

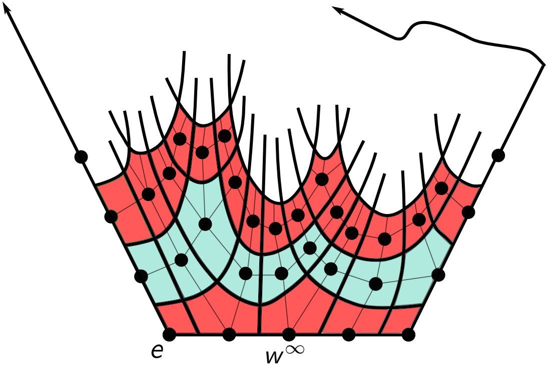


Figure: W_{Γ_3} has divergence at least cubic

Subsequent work for general W_Γ

Theorem (Behrstock–Hagen–Sisto)

1. *The divergence of W_Γ is either exponential (if the group is relatively hyperbolic) or bounded above by a polynomial (if the group is thick).*
2. *W_Γ has linear divergence $\iff$ Γ is a join.*

Behrstock, Falgas-Ravry, Hagen and Susse generalised the $\mathcal{CFS}$ condition to all Γ.

Theorem (Levcovitz)

1. *W_Γ has quadratic divergence $\iff$ Γ is $\mathcal{CFS}$.*
2. *If Γ contains a "rank d pair" and has "hypergraph index d", then W_Γ has divergence r^{d+1}.*