Very well-poised hypergeometric series and multiple integrals

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The purpose of this note is to establish a relationship between two objects: the very well-poised hypergeometric series

\[ F_k(h) = F_k(h_0; h_1, \ldots, h_k) := \sum_{\mu=0}^{\infty} \binom{h_0 + 2\mu}{\mu} \frac{\prod_{j=0}^{k} \Gamma(h_j + \mu)}{\prod_{j=0}^{k} \Gamma(1 + h_0 - h_j + \mu)} (-1)^{(k+1)\mu} \]

\[ = \frac{\Gamma(1 + h_0) \prod_{j=1}^{k} \Gamma(h_j)}{\prod_{j=1}^{k} \Gamma(1 + h_0 - h_j)} \Gamma(1 + h_0, 1 + \frac{1}{2} h_0, \ldots, h_k) \Gamma(1 + h_0 - h_1, \ldots, 1 + h_0 - h_k) \frac{(-1)^{k+1}}{(k+1)} \]

and the multiple integrals

\[ J_k(a, b) := J_k(\frac{a_0, a_1, \ldots, a_k}{b_1, \ldots, b_k}) := \int_{[0, 1]^k} \prod_{j=1}^{k} \frac{x_j^{a_j-1}(1 - x_j)^{b_j - a_j - 1}}{(1 - (1 - (\cdots (1 - (1 - x_{k-1})x_2)x_1)x_0)dx_1 dx_2 \cdots dx_k}. \]

**Theorem.** Suppose that \( k \geq 1 \), the parameters \( h_0, h_1, \ldots, h_{k+2} \in \mathbb{C} \) satisfy the conditions

\[ 1 + \Re h_0 > \frac{2}{k+1}, \quad \sum_{j=1}^{k+2} \Re h_j, \quad \Re(1 + h_0 - h_{j+1}) \geq \Re h_j > 0 \quad \text{for } j = 2, \ldots, k + 1, \]

and \( h_1, h_{k+2} \neq 0, -1, -2, \ldots \). Then the following identity holds:

\[ \frac{\prod_{j=1}^{k+1} \Gamma(1 + h_0 - h_j - h_{j+1})}{\Gamma(h_1) \Gamma(h_{k+2})} \cdot F_{k+2}(h_0; h_1, \ldots, h_{k+2}) = J_k(h_1, h_2, h_3, \ldots, h_{k+1}, 1 + h_0 - h_3, 1 + h_0 - h_4, \ldots, 1 + h_0 - h_{k+2}). \]

The proof is carried out by induction. If \( k = 1 \), then the statement of the theorem follows from the limit case of Dougall’s theorem ([1], §4.4, (1)). If \( k \geq 2 \), then we set \( \varepsilon_k = 0 \) for \( k \) even and \( \varepsilon_k = 1 \) or \(-1\) for \( k \) odd and use the relation

\[ J_k(\frac{a_0, a_1, \ldots, a_{k-1}, a_k}{b_1, \ldots, b_{k-1}, b_k}) = \frac{\Gamma(b_k - a_k)}{\Gamma(a_0)} \cdot \frac{1}{2\pi i} \int_{t_0}^{t_0 + i\infty} \Gamma(a_0 + t) \Gamma(a_k + t) \Gamma(-t) \Gamma(b_k + t) \]

\[ \times J_{k-1}(\frac{a_0 + t, a_1 + t, \ldots, a_{k-1} + t}{b_1 + t, \ldots, b_{k-1} + t}) dt, \]

where \( t_0 \in \mathbb{R}, \Re a_0 > t_0 > 0, \Re a_k > t_0 > 0, \Re b_k > \Re a_0 + \Re a_k \), provided that the integral on the left-hand side of (4) converges. Representing the hypergeometric series (1) in the form of a Barnes-type contour integral and applying the inductive hypothesis to the integrand on the right-hand side of (4), we obtain the desired identity (3).
We note that the series on the right-hand side of (3) admits a 'less economical' representation in the form of an Euler-type multiple integral over the cube \([0,1]^{k+2}\) (see [2], Lemma 1). The above theorem and recent results of Zlobin ([3], [4]) also yield a representation of the very well-poised hypergeometric series (1) in the form of an integral proposed in Sorokin’s papers [5], [6].

In spite of the analytic nature of the theorem, the identity (3) is motivated by arithmetic results for the values of the Riemann zeta function (zeta values) at positive integers ([5]–[13]). As is known [13], in the case of integral parameters \(h\) a very well-poised hypergeometric series of the form (1) is a \(Q\)-linear form in even or odd zeta values, depending on the parity of \(k \geq 4\). Therefore, if the parameters \(a\) and \(b\) are positive and integral and satisfy the additional condition

\[
b_1 + a_2 = b_2 + a_3 = \cdots = b_{k-1} + a_k,
\]

then the integral (2) is a \(Q\)-linear form in zeta values whose arguments are of the same parity. The specialization \(a_j = n+1\) and \(b_j = 2n+2\) leads to the coincidence (conjectured by the author in [13], §9) of multiple integrals and very well-poised hypergeometric series; denoting the corresponding integrals (2) by \(J_{k,n}\) and using the arithmetic results in [12], Lemmas 4.2–4.4, we conclude that

\[
D_n^{k+1} \Phi_n^{-1} \cdot J_{k,n} \in \mathbb{Z} \zeta(k) + \mathbb{Z} \zeta(k-2) + \cdots + \mathbb{Z} \zeta(3) + \mathbb{Z} \quad \text{for } k \text{ odd},
\]

where \(D_n\) is the least common multiple of the numbers \(1,2,\ldots,n\) and \(\Phi_n\) is the product of the primes \(p < n\) such that \(2/3 \leq \{n/p\} < 1\) (\(\{ \cdot \}\) stands for the fractional part of a number). The relations (6) (with the multiple \(D_n^{k+1}\) instead of \(D_n^{k+1} \Phi_n^{-1}\)) were conjectured by Vasil’ev [14] (see also [11], comment to Theorem 2) and proved by him for \(k = 5\) (the case \(k = 3\) was treated in [7]). Thus, we give a particular answer to Vasil’ev’s conjecture. The choice \(a_j = r n + 1\) and \(b_j = (r + 1)n + 2\) in (2) (or, equivalently, \(h_0 = (2r + 1)n + 2\) and \(h_j = r n + 1\) for \(j = 1,\ldots,k + 2\) in (1)) with an integer \(r \geq 1\) depending on a given odd integer \(k\) leads to the linear forms (in odd zeta values) similar to those considered by Rivoal [10] in the proof of his remarkable result that the sequence \(\zeta(3), \zeta(5), \zeta(7), \ldots\) contains infinitely many irrationals.

Moreover, it should be noted that if the assumption (5) holds, then the quantity

\[
\frac{F_{k+2}(h_0; h_1,\ldots,h_{k+2})}{\prod_{j=1}^{k+2} \Gamma(h_j)} = \frac{\prod_{j=1}^{k} \Gamma(a_j) \cdot \Gamma(b_j + a_2 - a_0 - a_1) \cdot \prod_{j=1}^{k} \Gamma(b_j - a_j)}{\prod_{j=1}^{k} \Gamma(a_j) \cdot \Gamma(b_1 + a_2 - a_0 - a_1) \cdot \prod_{j=1}^{k} \Gamma(b_j - a_j)}
\]

is obviously invariant under the action of the \((h\text{-trivial})\) group \(\Theta\) (of order \((k+2)!\)) consisting of all permutations of the parameters \(h_1,\ldots,h_{k+2}\). This result also has number-theoretic applications. For \(k = 2\) and \(k = 3\) the change of variables \((x_{k-1}, x_k) \mapsto (1 - x_k, 1 - x_{k-1})\) in (2) gives an additional transformation \(\epsilon\) of both the integral (2) and the series (1); for \(k \geq 4\) this transformation is not available, since the condition (5) is violated. The groups \((\Theta, \epsilon)\) of orders 120 and 1920 for \(k = 2\) and \(k = 3\), respectively, are known ([8], [9]); Rhin and Viola use these groups to obtain nice estimates for the irrationality measures of \(\zeta(2)\) and \(\zeta(3)\). For \(k \geq 4\) the group \(\Theta\) admits a natural interpretation as a permutation group of the parameters \(e_{1l} = h_l - 1, 1 \leq l \leq k + 2\), and \(e_{jl} = h_0 - h_j - h_l, 1 \leq j < l \leq k + 2\) (for details, see [13], §9).

Bibliography


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