On the irrationality of $\zeta_q(2)$

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For complex $q$, $|q| < 1$, we define the quantity

$$\zeta_q(2) := \sum_{n=1}^{\infty} \frac{q^n}{(1 - q^n)^2} = \sum_{n=1}^{\infty} \sigma(n)q^n; \quad \lim_{q \to 1^{-1}} (1 - q)^2 \zeta_q(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6},$$

where $\sigma(n)$ is the sum of divisors of the positive integer $n$.

**Theorem 1.** When $q = 1/p$, $p \in \mathbb{Z} \setminus \{0, \pm 1\}$, the number $\zeta_q(2)$ is irrational and its index of irrationality satisfies the inequality $\mu(\zeta_q(2)) \leq 4.07869374 \ldots$.

Recall that the index of irrationality $\mu(\alpha)$ of a number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is defined as the least upper bound of those $\mu \in \mathbb{R}$ for which the inequality $|\alpha - a/b| \leq |b|^{-\mu}$ has a finite number of solutions for $a, b \in \mathbb{Z}$. (Note that $\mu(\alpha) \geq 2$ by Dirichlet’s theorem.) If $\mu(\alpha) < +\infty$, we say that $\alpha$ is a Liouville number. A theorem of Nesterenko [1] implies the transcendence of $\zeta_q(2)$ for any $q \in \mathbb{Q}$ with $0 < |q| < 1$, although it does not follow from general bounds for the measure of transcendence [2] that this is a Liouville number.

We shall use standard $q$-notation [3]:

$$(T; q)_n := \prod_{k=1}^{n} (1 - q^{k-1}T), \quad \Gamma_q(t) := \frac{(q; q)_\infty (1 - q)^{1-t}}{(q^t; q)_\infty}, \quad [n]_q! := \Gamma_q(n + 1) = \frac{(q; q)_n}{(1 - q)^n}.$$

For each $n = 0, 1, 2, \ldots$ we define numbers $a_j = \alpha_j n + 1$, $j = 1, 2, 3$, $b_1 = \beta_1 n + 1$, $b_k = \beta_k n + 2$, $k = 2, 3$, where the integer parameters (directions) $\alpha_j$ and $\beta_1, \beta_k$ satisfy the conditions $\beta_1 = 0 \leq \alpha_j \leq \beta_k$, $\alpha_1 + \alpha_2 + \alpha_3 \leq \beta_1 \leq \beta_2 + \beta_3$. Consider the $q$-basic hypergeometric series [3]

$$H_n(q) := \sum_{t=0}^{\infty} \frac{[b_j - a_j - 1]![b_3 - a_3 - 1]q^t}{(1 - q)^{2[a_1 - b_1]q^t}} = \frac{\Gamma_q(t + a_1) \Gamma_q(t + a_2) \Gamma_q(t + a_3)}{\Gamma_q(t + b_1) \Gamma_q(t + b_2) \Gamma_q(t + b_3)} \cdot q^{t(b_2 + b_3 - a_1 - a_2 - a_3 - 1)}.$$

By decomposing $R(q; t)$ as a rational function of $T = q^t$ into a sum of partial fractions and performing the summation in (1), we arrive at the following assertion.

**Lemma 1.** $H_n(q) = A_n(q)\zeta_q(2) - B_n(q)$, where $A_n(q)$ and $B_n(q)$ are rational functions of the parameter $q$.

Explicit formulae for $A_n(q)$ and trivial estimates for the series on the right-hand side of (1) lead to the following result.

**Lemma 2.** For any $q = 1/p$, $p \in \mathbb{Z} \setminus \{0, \pm 1\}$,

$$\lim_{n \to \infty} \frac{\log |H_n(q)|}{n^2 \log |p|} = 0, \quad \lim_{n \to \infty} \frac{\log |A_n(q)|}{n^2 \log |p|} \leq \beta_2 \beta_3 - \frac{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}{2} =: C_1.$$

To calculate the denominators of the rational functions $A_n(q), B_n(q)$ as in [4], [5] for linear approximations to $\zeta(2)$, we apply a group $\mathcal{S} \subset \mathcal{S}_{10}$ of permutations of the 10-element set

$$c_{00} = (\beta_2 + \beta_3) - (\alpha_1 + \alpha_2 + \alpha_3), \quad c_{jk} = \begin{cases} \alpha_j - \beta_k & \text{if } k = 1, \\ \beta_k - \alpha_j & \text{if } k = 2, 3, \end{cases} \quad j, k = 1, 2, 3. \quad (2)$$

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This group has 120 elements, and the quantity

\[
H_n(q) = \frac{[c_{00} n!] [c_{21} n!] [c_{22} n!] [c_{33} n!] [c_{31} n!]}{[c_{00} n!] [c_{21} n!] [c_{22} n!] [c_{33} n!] [c_{31} n!]},
\]

is invariant under its action. Moreover, the quantity \( H_n(q) \) itself is invariant under the action of a subgroup \( \mathcal{G}_0 \subset \mathcal{G} \) of order 10. We put

\[
M := \max_{g \in \mathcal{G}_0} \{ \widetilde{M}(gc) \}, \quad \widetilde{M}(c) := \begin{cases} c_{00} c_{21} + c_{31} c_{33} - c_{21} c_{33} & \text{if } c_{21} \leq c_{31}, \\ c_{00} c_{31} + c_{21} c_{22} - c_{31} c_{22} & \text{if } c_{21} \geq c_{31}, \end{cases} \\
\omega(z) := \max_{g \in \mathcal{G}} (\tilde{\omega}(c; z) - \tilde{\omega}(gc; z)), \quad \tilde{\omega}(c; z) := [c_{00} z] + [c_{21} z] + [c_{22} z] + [c_{33} z] + [c_{31} z],
\]

where \( gc \) denotes the action of the corresponding permutation on the set (2), \( [\cdot] \) denotes the integer part function, and the function \( \omega(z) \) takes non-negative integer values and is 1-periodic. Also let \( m_1 \geq m_2 \) be two maximal elements standing in different places in the tuple \( c \). The cyclotomic polynomials \( \Phi_i(x) \), and only these, occur in the decomposition of \( (x; z)_n \) into irreducible factors (see, for example, [6], [7]), and the polynomial \( D_n(x) := \prod_{i=1}^n \Phi_i(x) \) is the least common multiple of \( x-1, x^2-1, \ldots, x^n-1 \).

**Lemma 3.** Let \( \Pi_n(p) := p^{-M n^2} \cdot D_{m_1 n}(p) D_{m_2 n}(p) \cdot \prod_{i=1}^{m_1 m_2} \Phi_i(p)^{-\omega(n/1)} \), where \( p = q^{-1} \in \mathbb{Z} \setminus \{0, \pm 1\} \). Then the coefficients of the linear form \( H_n(q) \) satisfy the inclusions \( \Pi_n(p) A_n(q) \), \( \Pi_n(p) B_n(q) \in \mathbb{Z} \).

To study the asymptotics of \( \Pi_n(p) \) as \( n \to \infty \) we apply the corresponding result [6] on the asymptotics of \( D_n(p) \) and the \( q \)-analogue of the arithmetic scheme of Chudnovskii–Rukhadze–Hata.

**Lemma 4.**

\[
- \lim_{n \to \infty} \frac{\log |\Pi_n(p)|}{n^2 \log |p|} = M - \frac{3}{\pi^2} \left( m_1^2 + m_2^2 + \int_0^1 \omega(z) \, d\psi'(z) \right) =: C_0,
\]

where \( \psi(z) \) is the logarithmic derivative of the gamma-function.

If \( C_0 > 0 \), then \( \zeta_q(2) \) is irrational for any \( q = 1/p, p \in \mathbb{Z} \setminus \{0, \pm 1\} \), and \( \mu(\zeta_q(2)) \leq C_1/C_0 \). Taking \( \alpha_1 = 5, \alpha_2 = 6, \alpha_3 = 7, \beta_2 = 14, \beta_3 = 15 \), we get \( C_0 = 38.00236293 \ldots \) and \( C_1 = 155 \), which yields the bound in Theorem 1.

The \( q \)-arithmetic scheme and the \( q \)-hypergeometric construction of approximating linear forms also enable us to sharpen the measures of irrationality [6], [7] for the quantities

\[
\zeta_q(1) = \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n}, \quad \ln_q(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^n}{1 - q^n}, \quad |q| < 1,
\]

which are the \( q \)-analogues of the (divergent) harmonic series and \( \log 2 \), respectively.

**Theorem 2.** For \( q = 1/p, p \in \mathbb{Z} \setminus \{0, \pm 1\} \), the indices of irrationality of the numbers (3) satisfy the inequalities \( \mu(\zeta_q(1)) \leq 2.49846482 \ldots \), \( \mu(\ln_q(2)) \leq 3.29727451 \ldots \).

**Bibliography**


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