On the irrationality of the values of the zeta function at odd integer points

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Only a few results are currently known on the arithmetic nature of the values of the Riemann zeta function $\zeta(s)$ for odd $s > 1$. The first of these is the irrationality of $\zeta(3)$, which was proved by Apery [1] in 1978; the most recent is the result of Rivoal [2], who established in 2000 the following asymptotic estimate for the dimensions $\delta(a)$ of the spaces spanned over $\mathbb{Q}$ by the numbers $1, \zeta(3), \zeta(5), \ldots, \zeta(a - 2), \zeta(a)$ for odd $a$:

$$
\delta(a) \geq \frac{\log a}{1 + \log 2} \left(1 + o(1)\right) \quad \text{as } a \to \infty. \quad (1)
$$

In particular, it follows from (1) that infinitely many of the values $\zeta(3), \zeta(5), \ldots$ are irrational. We generalize Rivoal’s construction from [2] and prove the following results.

**Theorem 1.** Each of the sets

$$
\{\zeta(5), \zeta(7), \zeta(9), \zeta(11), \zeta(13), \zeta(15), \zeta(17), \zeta(19), \zeta(21)\},
\{\zeta(7), \zeta(9), \zeta(11), \ldots, \zeta(35), \zeta(37)\},
\{\zeta(9), \zeta(11), \zeta(13), \ldots, \zeta(51), \zeta(53)\}
$$

contains at least one irrational number.

**Theorem 2.** There exists an odd integer $a \leq 145$ such that $1, \zeta(3), \text{ and } \zeta(a)$ are linearly independent over $\mathbb{Q}$.

**Theorem 3.** For every odd $a \geq 3$, the following absolute estimate holds:

$$
\delta(a) > 0.395 \log a > \frac{2}{3} \frac{\log a}{1 + \log 2}.
$$

We fix positive odd parameters $a, b$, and $c$ such that $a > b(c - 1)$ and $c \geq 3$, and for each positive $n$ we consider the rational function

$$
R(t) = R_n(t) := \frac{(t \pm (n + 1)) \cdots (t \pm cn)}{(t \pm 1) \cdots (t \pm n)} \cdot (2n)^{a+b-bc},
$$

$$
= (-1)^n \left( \frac{\Gamma(\pm t + cn + 1)}{\Gamma(\pm t + n + 1 + 1)} \right)^a \cdot \left( \frac{\Gamma(t)\Gamma(1-t)}{\Gamma(\pm t + n + 1)} \right)^a \cdot (2n)^{a+b-bc}, \quad (3)
$$

where the symbol $\pm$ means that the + and − signs both occur in the relevant product. On representing (3) as a sum of partial fractions, using the fact that it is odd, and recalling its behaviour as $t \to \infty$, we can conclude that

$$
I = I_n := \sum_{t=n+1}^{\infty} \frac{1}{(b-1)!} \frac{d^{b-1} R(t)}{dt^{b-1}} = \sum_{s \text{ odd} \atop b < s < a+b} A_s \zeta(s) - A_0, \quad (4)
$$

where the coefficients $A_s = A_{s,n}$ of the linear form $I$ are rational numbers. (When $b = 1$ and $c = 2r + 1$, we get the same linear forms (4) as in [2].) We denote by $D_n$ the least common multiple of $1, 2, \ldots, n$; as is well known,

$$
\lim_{n \to \infty} \frac{\log D_n}{n} = 1.
$$

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Lemma 1. For every odd integer \( c \geq 3 \) there exists a sequence of integers \( \Pi_n = \Pi_n(c) \geq 1 \), \( n = 1, 2, \ldots \), such that the numbers \( \Pi_n^{-b} P_{2n}^{a+b-1} A_{s,n} \) are integers and the following relation holds:

\[
\omega_c := \lim_{n \to \infty} \frac{\log \Pi_n(c)}{n} = - \sum_{t=1}^{(c-1)/2} \left( 2\psi \left( \frac{2l}{c-1} \right) + 2\psi \left( \frac{2l}{c} \right) + \frac{2c-1}{t} \right) + 2(c-1)(1-\gamma),
\]

in which \( \gamma \approx 0.57712 \) is Euler's constant and \( \psi(x) \) is the logarithmic derivative of the gamma function. (As \( c \to \infty \), the value of the quantity \( \omega_c \) in (5) is of order \( 2c(1-\gamma) + O(\log c) \).)

Lemma 1 strengthens the corresponding estimates for the denominators of the linear forms (4), at the expense of the appearance of the multipliers \( \Pi_n^{-b} \), even in the case \( b = 1 \) considered in [2]. This is pivotal in the deduction of Theorems 2 and 3.

The proof of the next assertion rests on the representation of the forms (4) as contour integrals and on application of the saddle-point method (cf. [4] and [5]). Additional restrictions are imposed on the parameters \( a, b, \) and \( c \) in the case where \( b > 1 \); they turn out to hold automatically in application to Theorem 1.

Lemma 2. The following limit relation holds for the linear forms (4):

\[
\sum_{n} \frac{\log |I_n|}{n} = \log \frac{2^{2a+b+c}\eta_0 + c!\eta_0 - c!bc}{|\eta_0 + 1|^{a+b}|\eta_0 - 1|^{a+b}},
\]

where \( \eta_0 \) is the real root of the polynomial \( (r + c)^{b(r - 1)^{a+b}} - (r - c)^{b(r + 1)^{a+b}} \) in the interval \( (c, +\infty) \) when \( b = 1 \), and it is one of the pair of complex conjugate roots with maximum possible value of \( \Re \eta_0 \) when \( b > 1 \). (For \( b = 1 \), the upper limit in (6) can be replaced by the ordinary limit, and then the value of \( \sum \) in (6) does not exceed \( (2a - c + 3) \log 2 - 2(a - c + 1) \log c \).)

By Lemmas 1 and 2, there is at least one irrational number among the values \( \zeta(s) \) for odd \( s \) such that \( b < s < a + b \), provided that \(-2b\omega_c + 2(a + b - 1) + \sum < 0 \). Theorem 1 now follows by taking the three sets of values \( a = 19, b = 3, c = 3; a = 33, b = 5, c = 3; a = 47, b = 7, c = 3 \) respectively for the alternatives in (2).

Lemma 3. The following estimate holds for the non-zero coefficients \( A_{s} = A_{s,n} \) of the linear forms (4):

\[
\lim_{n \to \infty} \frac{\log |A_{s,n}|}{n} \leq 2bc\log c + 2(a + b - bc)\log 2.
\]

Theorems 2 and 3 are deduced from Lemmas 1–3 using the linear independence criterion to be found in [6] in the same way as it was used in [2]. For the proof of Theorem 2 (where \( a = 145 \) and \( b = 1 \) we choose \( c = 21 \).)

Bibliography

[5] T. G. Khessani Plerud, Arithmetic properties of the values of hypergeometric functions, Candidate’s Dissertation, Moscow State University, Moscow 1999. (Russian)

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